Serrano's results on isotriviality criteria:

① If $\varphi: S \to C$ is an elliptic fibn, then $[\kappa(S, K_{S/C}) \leq 1$ $(\because K_{S/C} \hookrightarrow K_S)]$ φ is not isotrivial iff $\kappa(K_{S/C}) = 1$.

② Suppose $\varphi: S \to C$ is a fibration with general fiber of $g \geq 2$. If $\kappa(K_{S/C}) = 1$, then φ is isotrivial. Converse is 'almost true'.

Thm Let $\mathcal{F}$ be a reduced foliation with $\kappa = 1$. Then $\mathcal{F}$ is one of: (i) Riccati foliation (ii) Turbulent foliation, (iii) Non isotrivial elliptic fibration, (iv) Isotrivial fibn of $g \geq 2$.

Pf: $\kappa(\mathcal F) = 1 \Rightarrow \nu(\mathcal F) = 1$ & $P^2 = 0 \Rightarrow P$ is bpf, giving $\pi: X \to B$. If F is a general fiber of π, $K_{\mathcal F} \cdot F = 0$. Note that $K_{\mathcal F}^{\otimes m} = \mathcal O_X(nF + D)$ for some $m, n \in \mathbb N$ & $D \geq 0$ which is π-vertical.

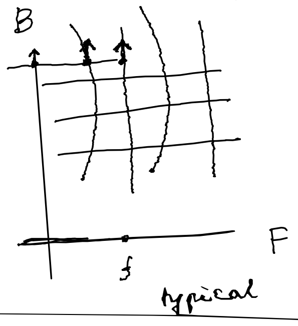
If $\mathcal F = T_{X|B}$, then $K_{\mathcal F} \cdot F = 0 \Rightarrow \pi$ is an elliptic fibn. Nonisotriviality follows from Serrano.

Otherwise $F \pitchfork \mathcal F$ b/c $\mathrm{tang}(\mathcal F, F) = K_{\mathcal F} \cdot F + F^2 = 0$. If F is rational or elliptic, $\mathcal F$ is Riccati or turbulent.

Suppose $X \xrightarrow{\pi} B$ with $g(F) \geq 2$; $F \pitchfork \mathcal F$, F the generic fiber of π. Let $B^* \subset B$ denote the $\mathcal F$-transverse fibers iso to F, in particular $\pi|_{X^*}: X^* \to B^*$ is a fiber bundle with fiber F. The monodromy $\rho: \pi_1(B^*) \to \mathrm{Aut}(F)$ is essentially trivial since $\mathrm{Aut}(F)$ is finite. Thus $\exists$ a finite base change $\nu: \tilde B \to B$ s.t. $\tilde\pi^*: \tilde X^* \to \tilde B^*$ is trivial with fiber F, i.e. $\tilde X^* \simeq F \times \tilde B^*$ [The category of locally constant sheaves on $\tilde B^*$ is equivalent to the category of $\pi_1(\tilde B^*)$-sets]. $\tilde{\mathcal F}|_{F \times \tilde B^*}$ is transverse to the fibers of $F \times \tilde B^* \to \tilde B^* \Rightarrow \tilde{\mathcal F}|_{\tilde X^*}$ is induced by $\tilde X^* \cong F \times \tilde B^* \to F$ (†). Thus the leaves of $\tilde{\mathcal F}$ are algebraic & are the closure of leaves of $\tilde{\mathcal F}|_{\tilde X^*}$, thus isom to $\tilde B$. $g(\tilde B) \geq 2$ ∴ $K_{\tilde{\mathcal F}} \cdot F = 0 \Rightarrow K_{\tilde{\mathcal F}} = (\tilde\pi)^* K_{\tilde B}$ ∴ $\kappa(K_{\tilde{\mathcal F}}) = 1$ [Alternatively $K_{\tilde{\mathcal F}} \cdot \tilde B > 0$ also shows this].

(†) Explanation: The projection $T_{F \times \tilde B^*} \xrightarrow{\pi} T_F$ restricts to give $\mathcal F \xrightarrow{\pi|_{\mathcal F}} T_F$. $\pi^{-1} t_f = t_f \times T_{(f \times \tilde B^*)}$

Suffices to check $\pi|_{\mathcal F}$ is an isom.



Last step is to check $\tilde{\mathcal F}: \tilde X \to F$ with fiber $\tilde B$ is an isotrivial fibration. But $\kappa(K_{\tilde X/F}) = \kappa(K_{\tilde{\mathcal F}}) = 1$ [Note: $K_{\tilde{\mathcal F}} = K_{\tilde X/F} - R$; R the ramification divisor. But R can be made 0 by (semi)stable reduction. ∴ $K_{\tilde{\mathcal F}} = K_{\tilde X/F}$]. Apply Serrano ②. □

Prop (Brun Ch 6) Suppose $h^0(X, N_F^*) \geq 1$ & that $\exists$ a non-contractible connected compact F-invariant curve $C \subset X$. Then F is a fibration over a curve with $g \geq 1$.

Pf: Let $0 \neq \omega \in H^0(N_F^*)$. Then (by F-inv of C), $\omega|_C \equiv 0$.
$\therefore$ Letting U denote a tubular nbd of C, the d-R coh class of $\omega|_U$ is zero. – ie $\omega = df$ for some holo fn f on U with $f|_C \equiv 0$. Note: the level sets of f are compact, hence algebraic. Non contractibility of $C \Rightarrow f$ vanishes only on C.
$\therefore$ f defines a proper fibr around C, hence a global one.

Thm Let F be a reduced foliation on X with $\kappa(F) = 0$. Then $\nu(F) = 0$.

Pf: Let $K_F = P + N = \sum_{j=1}^{p} a_j E_j + N$ be the Zariski decomp.
ETS: $P \equiv 0$.
Since birational reduced foliations have the same κ & ν, assume F is relatively minimal. Then Supp N is a disjt union of maximal F chains. Write $N = \sum_{j=1}^{m} N_j$, where $N_j = \sum_{i=1}^{k(j)} b_{j,i} D_{j,i}$ is a max F-chain for each j, $b_{j,i} \in (0,1)$. Clearly $P \cdot E_j = P \cdot D_{j,i} = 0$ $\forall$ i, j.

Claim: Each $E_j \subset \mathrm{Supp}(P)$ is F-invariant.
Pf: Suppose E_1 is not F-invariant. For each j, set

$$h(j) = \begin{cases} \min\{i = 1, \ldots, k(j) : D_{j,i} \cap E_1 \neq \phi\} & \text{if } N_j \cap E_1 \neq \phi \\ k(j)+1 & \text{if } N_j \cap E_1 = \phi. \end{cases}$$

Let $\overline{N_j} \doteq \sum_{i=h(j)}^{k(j)} b_{j,i} D_{j,i}$ (in particular $E_1 \cdot \overline{N_j} = E_1 \cdot N_j$).
& $Q \doteq E_1 + \sum_{j=1}^{m} \overline{N_j}$; note $\overline{N_j}$ is defd only when $E_1 \cap N_j \neq \phi$
Then $\forall$ $D_{j,i} \subset \mathrm{Supp}\, Q$, it's easy to check the following:

(i) $Q \cdot D_{j,h(j)} (= E_1 \cdot D_{j,h(j)} + N_j \cdot D_{j,h(j)} - b_{j,h(j)-1}) \geq 0$ & is > 0 unless $h(j) = 1$,

(ii) For $i > h(j)$, $Q \cdot D_{j,i} (= E_1 \cdot D_{j,i} + N_j \cdot D_{j,i}) \geq 0$ & > 0 unless $E_1 \cap D_{j,i} = \phi$.

(iii) $Q \cdot E_1 = (E_1 + N) \cdot E_1 = (E_1 + K_F) \cdot E_1 = \mathrm{tang}(F, E_1) \geq 0$ & > 0 unless $F \pitchfork E_1$.

(⊛Note $Q \cdot E_i = E_1 \cdot E_i + \sum \overline{N_j} \cdot E_i$ (for $i \geq 2$); $= 0$ $\therefore$ if $E_1 \cap E_i \neq \phi$, then $E_i \subset \mathrm{Supp}\, N$)

The intersection form is negative semidef on the v.s spanned by $\{E_j, D_{j,i}\}_{i,j}$, with kernel generated by P (recall $P \cdot E_j = P \cdot D_{j,i} = 0$ $\forall$ i, j. dim ker $= 1$ by HIT).
Thus $Q^2 \leq 0$. This (see inset) combined with (i), (ii), (iii) $\Rightarrow Q \cdot D_{j,i} = Q \cdot E_1 = 0$ $\forall$ i, j. ⊛Then Q is proportional to P by HIT.
We also conclude that $E_1 \pitchfork F$ & it intersects each N_j in atmost one point – on $D_{j,1}$. Since P & Q are proportional, $E_1 \cup \mathrm{Supp}(\sum_{j : N_j \cap E_1 \neq \phi} N_j)$ is a connected component of Supp P.

Let $X \to \hat{X}$ contraction of each maximal F-chain intersecting E_1. ($E_1 \subset X$, $\hat{E}_1 \subset \hat{X}$.) In a tubular nbd of $\hat{E}_1$, the leaves of $\hat{F}$ are discs intersecting $\hat{E}_1$ transversely. $s \in H^0(K_F^{\otimes l})$ gives a section $\hat{s} \in H^0(K_{\hat{F}}^{\otimes l})$ vanishing only on $\hat{E}_1$ when restricted to this tubular neighbourhood (b/c we've contracted all maximal F-chains intersecting E_1 & all components of P intersecting E_1 are actually components of N).
On each leaf (in this neighbourhood), $\hat{s}|_{\hat{L}} = df_{\hat{L}}$ for some holo fn f on $\hat{L}$ vanishing at the intn of $\hat{L}$ with $\hat{E}_1$. These fns patch to give a holo fn F on this nbd vanishing only along $\hat{E}_1$ (with some mult ≥ 2). As before, the level sets of F give a proper fibration on this nbd having $\hat{E}_1$ as a multiple fibre. This extends to a fibration on all of $\hat{X}$. But then $\kappa(P) = 1$ $\Rightarrow$⨳.
This finishes the proof of the claim. ▪

From now, assume Supp (P) is connected.

Def: A $\mathbb{Q}^+$-div $P = \sum_{j=1}^{p} a_j E_j$ is of elliptic fiber type if $P \cdot E_j = 0$ $\forall$ j & $K_X \cdot P = 0$.

Fact: Such divisors fall in Kodaira's list of singular fibers of elliptic fibrations.

Claim: The positive part P of K_F is of elliptic fiber type. Also Sing $F \cap \mathrm{Supp}(P) =$ Sing Supp (P) & all these singularities are non-degenerate.

Pf. $N_F \cdot E_j = E_j^2 + Z(F, E_j)$. Now $Z(F, E_j) \geq (\sum_{k \neq j} E_k) \cdot E_j$ (all cpts being invariant, each intersection contributes atleast 1 to $Z(F, E_j)$).
Thus $N_F \cdot E_j \geq (\sum E_k) \cdot E_j$, giving $N_F \cdot P \geq \sum E_k \cdot P = 0$.
Now, $K_X \cdot P = (N_F^* + P + N) \cdot P = N_F^* \cdot P$. If $N_F \cdot P > 0$, then Riemann–Roch $\Rightarrow \kappa(P) = 1$. Thus $N_F \cdot P = 0 = K_X \cdot P$. The claim about Singularities follows b/c all the inequalities above become equalities. ▪

Now we want to show $\exists$ an elliptic fibration with fiber P. This would contradict $\kappa(F) = 0$ showing $P \equiv 0$.

Case 1: $h^1(X, \mathcal{O}_X) > 0$.
Take $0 \neq \omega \in H^0(\Omega^1_X)$. If $\omega|_F \equiv 0$, it induces a non zero global section of N_F^* & then the above Propn $\Rightarrow F$ is an elliptic fibration with fiber P (as desired). If $\omega|_F \not\equiv 0$ $\forall$ $\omega \in H^0(\Omega^1_X)$, then each ω induces a non zero section of K_F. Now $h^0(K_F) \leq 1 \Rightarrow h^0(\Omega^1_X) = 1$. $\therefore$ $\mathrm{alb}_X : X \to B$ is a fibration over an elliptic curve. There are two possibilies:
a) P is a fiber of alb_X & we're done.
b) P is transverse to alb_X. Any non trivial 1-form anyway defines a section of K_F; transversality of $P \Rightarrow$ the section doesnot vanish along P; this contradicts $K_F^{\otimes l} = \mathcal{O}_X(l(P+N))$.

Case 2: $h^1(\mathcal{O}_X) = 0$.
Let Q be the smallest integral multiple of P. We claim that showing a multiple of Q is a fiber is equivalent to showing $\mathcal{O}_X(Q)|_Q$ is torsion. Indeed, let m be the order of $\mathcal{O}_X(Q)|_Q$. Considering exact sequences
$$0 \to \mathcal{O}_X((n-1)Q) \to \mathcal{O}_X(nQ) \to \mathcal{O}_X(nQ)|_Q \to 0$$
for $n = 1, \ldots, m-1$ gives us $h^1(\mathcal{O}_X(m-1)Q) = h^1(\mathcal{O}_X) = 0$ (recursively). $\therefore$ $h^0(\mathcal{O}_X(mQ)) = 2$ & hence Q is bpf as needed.

The next step is to show that $N_F^* \otimes \mathcal{O}_X(Q_{red}) = \mathcal{O}_X(-kQ)$ for some $k \in \mathbb{N}$ (in a nbd of Q) (*)
(We skip the proof)

St1: Now suppose Q is a smooth elliptic curve. Then $N_F|_Q = \mathcal{O}_X(Q)$ & (*) shows that $\mathcal{O}_X(Q)|_Q$ is torsion.

St2: Similar analysis can be applied if Q is a cycle of (-2) curves.

General case: Q is of elliptic fiber type $\therefore$ it's in Kodaira's list of singular fibers of elliptic fibrations. Suppose Q is of type II, ie $Q = 6E + 3D_1 + 2D_2 + D_3$ where E, D_i are smooth ratl curves with $E^2 = -1$, $D_1^2 = -2$, $D_2^2 = -3$, $D_3^2 = -6$.

Idea: though Q is not a fiber, we can apply "Stable Reduction" to it: $\exists$ $W \xleftarrow{r} V \xrightarrow{\pi} X$ where π is a finite cover extracting "6th root of Q" & r is a blow down of rational curves in $\pi^{-1}(Q)$. We have:
(i) $C = r(\pi^{-1}(Q_{red}))$ is smooth elliptic curve of zero self-intersection & $r_* \pi^* \mathcal{O}_X(Q) = \mathcal{O}_W(6C)$; (ii) $\mathcal{G} \doteq r_* \pi^* F$ is tangent to C & smooth around it (OK since all sings arose from intersections of comps of Q);
(iii) $N_{\mathcal{G}}^* \otimes \mathcal{O}_W(C) = r_* \pi^* (N_F^* \otimes \mathcal{O}_X(Q_{red})) = \mathcal{O}_W(-6kC)$.

Now to prove $\mathcal{O}_X(Q)|_Q$ is torsion, NTS: $\mathcal{O}_W(6C)|_{6C}$ is torsion ie $\mathcal{O}_W(C)|_{6C}$ is torsion.
For this, we need an analogue of (*):
on a nbd of C, $N_{\mathcal{G}}^* \otimes \mathcal{O}_W(C) = \mathcal{O}_W(-hC)$ for some $h \in \mathbb{N} \setminus 6\mathbb{N}$ (†). Proof skipped.
With this, we're done: (iii) $\Rightarrow \mathcal{O}_W(-6kC) = \mathcal{O}_W(-hC)$. Since $6 \nmid h$, $\mathcal{O}_W(C)$ is torsion.
$\therefore$ $\mathcal{O}_W(C)|_{6C}$ is also torsion.

We thus conclude that Q is a fiber of an elliptic fibn. This contradiction $\Rightarrow P = 0$. ▪

Cor: Let F be a reduced foliation with $\kappa(F) \neq \nu(F)$. Then $\kappa(F) = -\infty$ & $\nu(F) = 1$.

Pf: $\kappa(F) = 0 \Leftrightarrow \nu(F) = 0$. $\kappa(F) = 1 \Rightarrow P$ is bpf & $P^2 = 0$ $\therefore$ $\nu(F) = 1$. $\kappa(F) = 2 \Leftrightarrow \nu(F) = 2$. ▪

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