$\mathcal{F}$ foliation on X smooth surface.

$N_{\mathcal{F}}$ = normal bundle to $\mathcal{F}$.

Take $p \in X \setminus \mathrm{Sing}(\mathcal{F})$ Choose $(\cancel{z}\, z_1, z_2)$ local coord. at p.

$\Rightarrow N^*_{\mathcal{F}} = (dz_1)$ [or equivalently by any $\omega = f(z_1, z_2)\, dz_1$
$w/\ f(0,0) \neq 0$; ω is not closed but
$d\omega = \beta \wedge \omega$, where $\beta := \frac{df}{f}$]

Take $p \in \mathrm{Sing}(\mathcal{F})$. If ω generates $N^*_{\mathcal{F}}$ at p, $d\omega(p)$ could be $\neq 0$. [e.g., $\mathcal{F} = (v)$, $v = x\partial_x \mp y\partial_y$
$\Rightarrow \omega = y\,dx \mp x\,dy \Rightarrow d\omega = 2dx \wedge dy$]

$\Longrightarrow$ it is not possible ~~fo~~ to write $d\omega = \beta \wedge \omega$ $w/\ \beta$ holomorphic of type $(1,0)$.

However, we can do that if we accept to use smooth forms.

If we write $\omega = A(z_1, z_2)\, dz_2 - B(z_1, z_2)\, dz_1$,
$A(0,0) = B(0,0) = 0$ A, B holomorphic, then

$$\beta := F(z_1, z_2) \frac{\partial_{z_1} A + \partial_{z_2} B}{|A|^2 + |B|^2} \left(\bar{A}\, dz_1 + \bar{B}\, dz_2 \right)$$

with F 
p

$\Rightarrow$ • $\exists$ an open covering $\{U_j\}_{j \in J}$ of X,

• $\omega_j \in \Omega^1_X(U_j)$ w/ isolated 0's which generate $N^*_{\mathcal{F}}|_{U_j}$

• $\beta_j \in A^{1,0}(U_j)$

such that $\forall j \in J$

$\exists\ V_j \subsetneq U_j$ with $V_j \cap U_{j'} = \emptyset$ $\forall j' \in J \setminus \{j\}$ and
$d\omega_j = \beta_j \wedge \omega_j$ on $U_j \setminus V_j$.

In particular, $d\omega_j = \beta_j \wedge \omega_j$ on $U_j \cap U_{j'}$ $\forall j' \neq j$.

Also, on $U_j \cap U_{j'}$ $\omega_{j'} = g_{jj'}\, \omega_j$ in terms of cocycles.

$\Rightarrow d\omega_{j'} = \beta_{j'} \wedge \omega_{j'} = dg_{jj'} \wedge \omega_j + g_{jj'}\, d\omega_j = dg_{jj'} \wedge \frac{\omega_{j'}}{g_{jj'}} +$

$\omega_{j'} \wedge \left(\frac{dg_{jj'}}{g_{jj'}} + \beta_j \right) = + g_{jj'} (\beta_j \wedge \omega_j)$

$\Rightarrow \omega_j' \wedge \left(\frac{dg_{jj'}}{g_{jj'}} + \beta_j - \beta_j' \right) = 0$ on $U_j' \cap U_j$.

$\frac{dg_{j'j}}{g_{j'j}} + \beta_j - \beta_j'$ gives a cocycle of smooth sections of $N^*_{\mathcal{F}}$.

$\Rightarrow$ Since $A^{\infty}(N^*_{\mathcal{F}})$ has trivial cohomology, being fine (as C^∞-module)

$\Rightarrow$ $\forall j \in J$ $\exists$ $\gamma_j \in A^{1,0}(U_j)$ s.t.

- $\gamma_j \wedge \omega_j \equiv 0$ on U_j
- $\gamma_{j'} - \gamma_j = \frac{dg_{jj'}}{g_{jj'}} + \beta_j - \beta_j'$ on $U_j \cap U_{j'}$ (*)

On $U_j \setminus V_j$ we have $d\omega_j = (\beta_j + \gamma_j) \wedge \omega_j$

Define $\Omega := \frac{1}{2\pi i} d(\beta_j + \gamma_j)$ on U_j.

$$\Omega = [c_1(N_{\mathcal{F}})] \text{ in } H^2(X, \mathbb{R})$$

By (*) $[\Omega] = \left[\frac{1}{2\pi i}\partial\bar\partial \log(g_{ij})\right]$.

This is not representation in $H^{1,1}_{\mathcal{F}}$, for that we should consider $\Omega_{\bar\partial} = \frac{1}{2\pi i}\bar\partial(\beta_j + \gamma_j)$.

$\forall p \in \operatorname{Sing} \mathcal{F}$, we now define: $[\omega = A\,dw - B\,dz$ around $p]$

$$BB(\mathcal{F}, p) = \operatorname{Res}_{(0,0)} \left\{ \frac{[A_z(z,w) + B_w(z,w)]^2}{A(z,w)\,B(z,w)} dz \wedge dw \right\}$$

$$BB(\mathcal{F}, p) = \frac{1}{(2\pi i)^2} \int_{B(p,\varepsilon)} \beta \wedge d\beta$$

When p is non-degenerate $\Rightarrow$ if λ_i are the eigenvalues then

$$BB(\mathcal{F}, p) = \frac{(\lambda_1 + \lambda_2)^2}{\lambda_1 \lambda_2} = \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1} + 2.$$

THM $N_{\mathcal{F}}^2 = \sum_{p\in Sing(\mathcal{F})} BB(\mathcal{F},p)$

Proof $N_{\mathcal{F}}^2 = \int_X \Omega\wedge\Omega$. But outside the V_j's (around each $p\in Sing(\mathcal{F})$)

$[\Omega] = [\frac{1}{2\pi i} d\tilde\beta_j]$ and $\tilde\beta_j \wedge \omega_j = d\omega_j \Rightarrow \Omega\wedge\omega_j = \frac{1}{2\pi i} d\tilde\beta_j \wedge \omega_j$

$$0 = d^2\omega_j = d\tilde\beta_j\wedge\omega_j - \tilde\beta_j\wedge d\omega_j = d\tilde\beta_j\wedge\omega_j - \cancel{\tilde\beta_j\wedge\tilde\beta_j\wedge\omega_j} = d\tilde\beta_j\wedge\omega_j$$

$$\Omega\wedge\Omega = d\tilde\beta_j\wedge d\tilde\beta_j = d(d\tilde\beta_j\wedge\tilde\beta_j) + d^2\tilde\beta_j\wedge\tilde\beta_j$$

$$\Rightarrow [\text{Stokes}] \int_X \Omega\wedge\Omega = \frac{1}{(2\pi i)^2}\int_{\partial V_j}\tilde\beta_j\wedge d\tilde\beta_j \quad \blacksquare$$

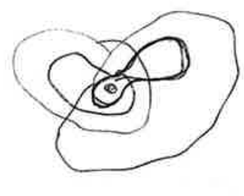
CAMACHO – SAD FORMULA

Notation as before. Now, also $C\subset X$ invariant curve.

$\forall p\in C\cap Sing(\mathcal{F})$, we can write

$$g\omega = h\,df + f\eta, \quad \eta \text{ hol. 1-form}, \quad g,h \text{ hol. fcts, prime to } f.$$

$$CS(\mathcal{F},C,p) = Res_p\left\{-\frac{1}{h}\eta|_C\right\} = -\frac{1}{2\pi i}\int_\gamma \frac{1}{h}\eta$$



$$Var(\mathcal{F},C,p) = \frac{1}{2\pi i}\int_\gamma \beta$$

PROPERTY $Var(\mathcal{F},C,p) = Z(\mathcal{F},C,p) + CS(\mathcal{F},C,p)$

$$Z(\mathcal{F},C,p) = ord_p\left(\frac{h}{g}\Big|_C\right) = \frac{1}{2\pi i}\int_\gamma \frac{d(\frac{h}{g})}{h/g}$$

But $\omega = \frac{h}{g}df + f\frac{\eta}{g}$. Defining $\beta_0 = \frac{d(\frac{h}{g})}{h/g} - \frac{1}{h}\eta$

$$\Rightarrow d\omega = \beta_0\wedge\omega + f\Theta, \quad \Theta\in\Omega^2_X(\text{nbd of }\gamma)$$

$$\Rightarrow \beta_0|_\gamma = \beta|_\gamma \quad \forall\beta \text{ s.t. } d\omega = \beta\wedge\omega \Rightarrow \checkmark$$

<u>THM</u> $$C^2 = \sum_{p \in C \cap \mathrm{Sing}(\mathcal{F})} CS(\mathcal{F}, C, p)$$

<u>Pf</u> <u>Before</u> $C^2 = N_{\mathcal{F}} \cdot C - \sum Z(\mathcal{F}, C, p)$

$N_{\mathcal{F}} \cdot C = \int_C \Omega$. By $\Omega \wedge \omega_j = 0$ on $U_j \setminus V_j$ we get

$$\sum \int_{V_j \cap C} \Omega|_{V_j} = \int_{\gamma_j} \tilde{\beta}_j = Z\,\mathrm{Var}(\mathcal{F}, C, p) \qquad \boxtimes$$