NUMERICAL DIMENSION OF FOLIATIONS

§ Introduction

X smooth proj. surface, L line bundle on X
Cartier divisor

Recall: The Iitaka-Kodaira dimension $\kappa(L) = \limsup_{n \to +\infty} \frac{\log h^0(X, nL)}{\log n} \in \{-\infty, 0, 1, 2\}$

Zariski-Fujita: If L is pseudoeffective (psef), i.e. in the boundary of effective divisors ($\Leftrightarrow L \cdot H \geq 0 \;\; \forall H$ ample) then $\exists$ $\mathbb{Q}$-divisors P and N with $L \overset{num}{\equiv} P + N$ s.t.

i) P is nef

ii) $N = \sum b_j D_j$ is effective and the intersection matrix $(D_i \cdot D_j)_{i,j}$ is negative definite

iii) ~~P~~ $\cancel{P \cdot N = 0}$ $P \cdot N = 0$

Moreover, N is unique. Since P is nef, $P^2 \geq 0$.

Numerical dimension of L:

$$\nu(L) = \begin{cases} -\infty & \text{if } L \text{ is not psef} \\ 0 & \text{if } P \equiv 0 \\ 1 & \text{if } P \not\equiv 0 \text{ but } P \cdot P = 0 \\ 2 & \text{if } P \cdot P > 0 \end{cases}$$

Def. L is called *abundant* if $\nu(L) = \kappa(L)$.

~~Ge~~ Facts:
- $\kappa(L) \leq \nu(L)$
- if $\nu(L) = 2$, then $\kappa(L) = 2$ (RR)

Thm K_X nef $\overset{\dim = 2,3}{\Rightarrow}$ K_X abundant $\overset{\text{Kawamata (any dimension)}}{\Leftrightarrow}$ K_X semiample

Let $\mathcal{F}$ on X be a foliation with reduced singularities.

$K_{\mathcal{F}} := T^*_{\mathcal{F}}$. Define $\kappa(\mathcal{F}) := \kappa(K_{\mathcal{F}})$ and $\nu(\mathcal{F}) := \nu(K_{\mathcal{F}})$

Lemma Let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be birational foliations with reduced singularities.

~~Assume they are birati~~ Then

$$\kappa(\mathcal{F}) = \kappa(\mathcal{G}) \quad \text{and} \quad \nu(\mathcal{F}) = \nu(\mathcal{G})$$

Proof We can assume $\mu : X \to Y$ is the blow-up of a point and $\mathcal{F}$ is induced by $\mathcal{G}$. Then $K_{\mathcal{F}} = \mu^* K_{\mathcal{G}} + E$ with E eff. exc (because reduced singularities). ← definition of canonical singularities

Here $\cancel{\kappa(\mathcal{F})} = h^0(nK_{\mathcal{F}}) = h^0(nK_{\mathcal{G}}) \quad \forall n \geq 0$.

Since ~~If~~ $K_{\mathcal{G}} \equiv P + N$ Zariski decomposition, then

$K_{\mathcal{F}} \equiv \mu^* P + N + E$ " " , so

$\nu(K_{\mathcal{F}}) = \nu(K_{\mathcal{G}})$. □

Situation: (recall $\kappa(X, \mathcal{F}) \leq \nu(\mathcal{F})$)

(⇒) Miyaoka

- $\nu(K_{\mathcal{F}}) = -\infty \Leftrightarrow \exists H$ ample s.t. $K_{\mathcal{F}} \cdot H < 0 \Leftrightarrow \mathcal{F}$ is given by a rational fibration (blowing-up)
  $\Downarrow$
  $\kappa(\mathcal{F}) = -\infty$

- if $\nu(K_{\mathcal{F}}) = 2 \overset{(\Leftarrow)}{\Rightarrow}$ ~~then~~ $\kappa(\mathcal{F}) = 2$ general type

- Today: $\nu(\mathcal{F}) = 0 \Rightarrow \kappa(\mathcal{F}) = 0$

- Spoiler: $\nu(\mathcal{F}) = 1 \Rightarrow \kappa(\mathcal{F}) = 1$ or $\kappa(\mathcal{F}) = -\infty$
  ↑
  Hilbert modular foliations

§ Numerical dimension 0 [Br, 8.1] X smooth proj surface

Thm 1 (McQuillan) Let $\mathcal{F}$ be a relatively minimal foliation on X s.t $K_{\mathcal{F}}$ is pseff and let $K_{\mathcal{F}} \equiv P + N$ be its Zariski decomp. Then ~~N~~ $\mathrm{Supp}(N)$ is a disjoint union of maximal $\mathcal{F}$-chains, in particular it is $\mathcal{F}$-invariant. Moreover, if $N = \sum b_j D_j$, then $0 < b_j < 1$.

Hirzebruch - Jung strings

$C = \bigcup C_i$ $C_i \cong \mathbb{P}^1$ $C_i^2 \leq -2$ contractible by ~~Grauert~~ Artin to a cyclic quotient singularity

Proof Maybe another time. It uses the separatrix theorem due to Camacho - Sad (through any singular point of $\mathcal{F}$ there exists at least one invariant curve). Roberto will talk about this. □

Thm 2 (McQuillan) [Br, 8.2] Let $\mathcal{F}$ be a relatively minimal (reduced) foliation on X s.t. $\nu(\mathcal{F}) = 0$. Then there exist $\pi : Y \to X$ generically finite (ramified covering) and a birational morphism $p: Y \to Z$ s.t. $p_* \pi^* \mathcal{F}$ is generated by a global holomorphic vector field with isolated zeroes

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & X \\ \downarrow p & & \\ Z & & \end{array}$$

Y, Z smooth

p ← contraction of $\mathcal{F}$-exc. curves (−1-curves $\mathcal{F}$ invariant

In particular, $\kappa(\mathcal{F}) = 0$

Remarks

1) $\mathcal{F}$ generated by a global vector field $\Leftrightarrow$ $\mathcal{F}$ corresponds to $s \in H^0(X, T_X)$

$\Leftrightarrow$ $K_{\mathcal{F}} \sim \mathcal{O}_X$

generated by global vector fields with isolated zeros

2) Foliations ~~with $\nu(\mathcal{F}) = 0$~~ are classified in 4 families: (Chapter 6 of Brunella)

(i) almost elliptic fibre bundles:

~~$\kappa(X) = 1$~~ $\pi: X \to B$ elliptic fibration, $\mathcal{F}$ given by a vector field with no zeroes and ~~tantey~~ tangent to the fibres of π.

$\kappa(X) = 0, 1$

Outside multiple fibres, π is a locally trivial fibration.

Ex: $E \times C \to C$

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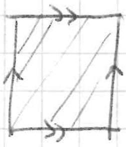
(ii) Kronecker foliations on tori. (linear) $k(X)=0$

Ex.: $X = \mathbb{C}^2/\mathbb{Z}^4$, $\mathcal{F}_\lambda = \{x + \lambda y = t\}$ family of foliations $\lambda \in \mathbb{R}$



the leaves are algebraic $\Leftrightarrow \lambda \in \mathbb{Q}$
(elliptic curves)
otherwise they fill up X
are not
compact and

(iii) suspensions of representations. $\rho: \pi_1(E) \to \mathrm{Aut}(\mathbb{P}^1)$, (monodromy)
E elliptic curve, i.e. we have
alb: $X \to E$, $k(X) = -\infty$ v generates a Riccati foliation
without invariant fibres, X minimal ($\mathbb{P}^1$-bundle)

(iv) up to birational maps, foliations on $X = \mathbb{P}^1 \times \mathbb{P}^1$ generated
$k(X) = -\infty$ by $v_1 \oplus v_2$, v_j hol. vector fields on $\mathbb{P}^1$

~~Sketch of~~

Proof of Thm 2.

By assumption $K_{\mathcal{F}} \equiv N$, where $\overset{\sum b_j D_j}{\overset{\shortparallel}{N}}$ is an effective contractible $\mathbb{Q}$-divisor which is $\mathcal{F}$-invariant and $b_j < 1$ by Thm 1.

Lemma ~~Step~~ **1.** If $h^0(X, K_{\mathcal{F}}) > 0$, then $K_{\mathcal{F}} \sim \mathcal{O}_X$.

Proof Let $s \in H^0(X, K_{\mathcal{F}}) \setminus \{0\}$. Then $K_{\mathcal{F}} \sim (s)_0 = N$, since N is unique in the Zariski decomposition of $K_{\mathcal{F}}$.

Since $(s)_0$ is integral, $N = 0$ ~~(so $K_{\mathcal{F}} \equiv 0$ and $K_{\mathcal{F}} \geq 0$)~~

$\Rightarrow K_{\mathcal{F}} \sim \mathcal{O}_X$. □

So in this case $\mathcal{F}$ corresponds to a global ~~section~~ vector field.

Lemma 2 There exists $n \in \mathbb{N}^+$ s.t. $h^0(X, nK_{\mathcal{F}}) > 0$. In particular $\kappa(\mathcal{F}) = 0$.

Proof ~~If case~~ If $h^1(X, \mathcal{O}_X) = 0$, then $\exists \overset{\mathbb{N}^+}{n}$ s.t. $nP \sim 0$ ($\equiv$ and $\sim$ coincide)

(nP integral and $\equiv 0$). Hence $nK_{\mathcal{F}} \sim nN$ effective, i.e. $h^0(X, nK_{\mathcal{F}}) > 0$.

So assume $\boxed{h^1(X, \mathcal{O}_X) \neq 0.}$ By Hodge theory, $H^1(X, \mathcal{O}_X) \simeq H^0(X, \Omega^1_X)$

If $\exists\, \omega \in H^0(X, \Omega^1_X)$ s.t. $\omega_{|\mathcal{F}}$ is not identically zero, then $H^0(X, K_{\mathcal{F}}) \neq 0$.

In fact, $\alpha: \mathcal{O}_X \xrightarrow{\omega} \Omega^1_X \underset{\text{def. of } \mathcal{F}}{\to} I_Z K_{\mathcal{F}} \hookrightarrow K_{\mathcal{F}}$ is non-zero

~~$\omega_{|\mathcal{F}}$ is non-zero~~, so we have a global section of $K_{\mathcal{F}}$ and we are done by Lemma 1.

Hence we can assume $\boxed{h^1(X, \mathcal{O}_X) \neq 0}$ and that $\boxed{\forall \omega \in H^0(X, \Omega^1_X)}$ $\boxed{\omega_{|\mathcal{F}} \equiv 0}$.

Consider the Albanese map $alb : X \to H^0(X, \Omega^1_X)^* / H_1(X, \mathbb{Z})$

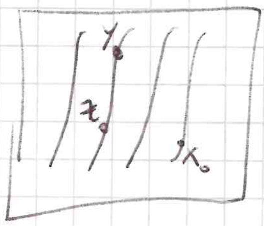
~~Fix $x_0 \in X$~~

$x \mapsto \int_{x_0}^{x} \cdot$ $\qquad x_0 \in X$ fixed

It is not trivial, since $H^0(X, \Omega^1_X)^* \neq 0$

We want to prove that $\mathcal{F}$ coincides with alb.

X



Take y, z in the same leaf, then we can take a path γ_1 from x_0 to z and then a path γ_2 from z to y, (at contained in ~~least~~ least if y and z are close enough)

Then $\forall \omega \in H^0(X, \Omega^1_X)$ $\qquad \int_{x_0}^{y} \omega = \int_{x_0}^{z} \omega + \underbrace{\int_{z}^{y} \omega}_{= 0}$.

so $alb(y) = alb(z)$

$\Rightarrow$ alb is a fibration in curves and the leaves of $\mathcal{F}$ are contained in the fibres of alb $\Rightarrow$ $\mathcal{F}$ is given by alb

$K_{\mathcal{F}} \equiv P + N$, N $\mathcal{F}$ invariant ($\Rightarrow N \subseteq$ fibres), so if F is a general fibre of alb

$$\underbrace{K_{\mathcal{F}} \cdot F}_{= 2g(F) - 2} = \underbrace{P \cdot F}_{= 0,\ P \equiv 0} + \underbrace{N \cdot F}_{= 0}$$

$\Rightarrow g(F) = 1$ so we have an elliptic fibration.

Using Kodaira's bundle formula, one can check that $\mathcal{F}$ is isotrivial.
(Serrano 1992 looks at $k(K_{X/B})$) $\qquad K_{\mathcal{F}} = K_{X/B}$

This implies that up to a finite cover, $X \to B$ is birational to a product. This proves the lemma and actually the theorem in the case $h^1(X, \mathcal{O}_X) \neq 0$ and $\omega_{|\mathcal{F}} \equiv 0$ $\forall \omega \in H^0(X, \Omega^1_X)$.

□

Hence, we can assume $\boxed{h^0(X, nK_{\mathcal{F}}) > 0}$ for some $n > 0$.

We use a cyclic covering trick. Take $s \in H^0(X, nK_{\mathcal{F}}) \setminus \{0\}$.

Then we get a $n:1$ cover $\Phi: Y_0 \to X$ ramified along $\operatorname{supp} N = \operatorname{supp}(s)_0$

~~Since supp N is SNC~~, We can resolve the singularities of Y_0,

$\pi: Y \to Y_0 \to X$ (*). Set $\mathcal{G} = \pi^* \mathcal{F}$

Since N is $\mathcal{F}$-invariant, we can check that $K_{\mathcal{G}} = \pi^* K_{\mathcal{F}}$.

$$H^0(Y, K_{\mathcal{G}}) = H^0(Y, \pi^* K_{\mathcal{F}}) = H^0(X, K_{\mathcal{F}} \otimes \pi_* \mathcal{O}_Y) = H^0\Big(X, K_{\mathcal{F}} \otimes \bigoplus_{j=0}^{n-1} K_{\mathcal{F}}^{-j}\Big)$$

So we get a section $(j=1)$

$K_{\mathcal{G}} = \pi^* P + \pi^* N$ Zariski decomposition $\Rightarrow \nu(\mathcal{G}) = 0$

Maybe $\mathcal{G}$ is not rel. minimal, we take the rel. minimal model

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & X \\ {\scriptstyle p}\downarrow & & \\ Z & & \end{array}$$

and we are done by Lemma 1. □

(*) Since $\operatorname{Supp} N$ is SNC, the singularities of the cover are of Hirzebruch-Jung type, so in the resolution $Y \to Y_0$ we only have Hirzebruch-Jung strings and (-1)-curves, all of them $\mathcal{G}$-invariant.

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§ Contraction of the negative part

Assume $K_{\mathcal{F}} = P + N$ psett. ($\mathcal{F}$ reduced singularities on a smooth s. relatively minimal. Then ~~N is a disjoint union~~ each ~~component~~ of N can be contracted to a cyclic quotient singularities and we obtain a normal surface X_0 with a foliation $\mathcal{F}_0$ s.t. $K_{\mathcal{F}_0}$ is nef (it is only a $\mathbb{Q}$-~~divisor~~ Cartier divisor)

The singularities of $(X_0, \mathcal{F}_0)$ belong to the class of canonical singularities and $(X_0, \mathcal{F}_0)$ is called a nef model.

Def A foliation $\mathcal{F}$ on a normal surface X is said to have canonical singularities if for any birational morphism $\tilde{X} \to X$, $\tilde{X}$ smooth, we have $K_{\tilde{\mathcal{F}}} \equiv \pi^* K_{\mathcal{F}} + R$ where $\tilde{\mathcal{F}} = \pi^* \mathcal{F}$ and R is an eff. exc. divisor.