§1. (Recap)

Recall: • definition of $\mathcal{F}$-exc curves.

• Facts about $\mathcal{F}$-exc curves: An $\mathcal{F}$-exc curve $C \subset X$ can be one of two kinds.

i) C contains only one singularity q of $\mathcal{F}$ with $Z(\mathcal{F}, C, q) = 1$. If $\pi: \underset{C}{X} \to \underset{p}{X'}$ is the blowdown of C, then $\pi(C) = p \in \mathcal{F}'_{sm}$.

ii) C contains two singular points q_1, q_2 of $\mathcal{F}$ with $Z(\mathcal{F}, C, q_1) = Z(\mathcal{F}, C, q_2) = 1$. Then p is a reduced singularity of $\mathcal{F}'$ & $a(p) = 1$.

• definition & existence of relatively minimal models, Minimal models. (Note: if $(X, \mathcal{F})$ is minimal, any birat'l self map is an isom)

• Prop: Let $(X, \mathcal{F})$ be a reduced foliation. Then

(i) $(X, \mathcal{F})$ is relatively minimal iff any birational morphism $(X, \mathcal{F}) \xrightarrow{\pi} (Y, \mathcal{G})$ onto a reduced foliation is an iso.

(ii) $(X, \mathcal{F})$ is minimal iff any birational map $(Y, \mathcal{G}) \dashrightarrow (X, \mathcal{F})$ from a reduced foliation is a morphism.

Examples: ① Let $\pi: X \to B$ be a rational fibration, $\mathcal{F} = T_{X/B}$. Then $\mathcal{F}$ is reduced & it's relatively minimal iff all fibers are smooth. $\mathcal{F}$ is never minimal [flipping the fiber].

[Recall: Riccati foliation: Let $\pi: X \to B$ be a rational fibration. A foliation $\mathcal{F}$ on X is Riccati wrt π if it's transverse to a general fiber of π.

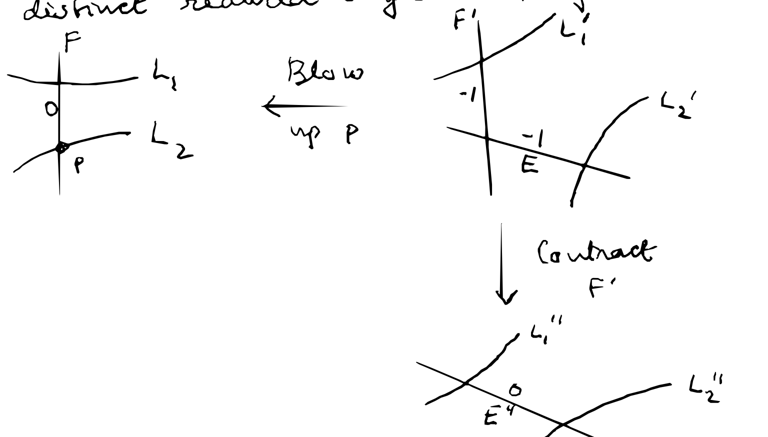
Prop: Let $\mathcal{F}$ be a foliation on X & $F \simeq \mathbb{P}^1 \subset X$ s.t. $F^2 = 0$ & F is $\mathcal{F}$-invariant. Then $\exists$ a rational fibn $\pi: X \to B$ with fiber F. We then have:

(i) If $Z(\mathcal{F}, F) = 0$, then $\mathcal{F} = T_{X/B}$.

(ii) If $Z(\mathcal{F}, F) = 2$, $\mathcal{F}$ is Riccati wrt π.]

(iii) $Z(\mathcal{F}, F) \neq 1$.

② Let $\mathcal{F}$ be Riccati wrt $\pi: X \to B$. Let $F \overset{\mathbb{P}^1}{=}$ be a smooth fiber of π which is $\mathcal{F}$-invariant & contains two distinct reduced singularities of $\mathcal{F}$.



If $(X, \mathcal{F})$ is relatively minimal, so is $(X'', \mathcal{F}'')$. But X & X'' can't be isom, $\therefore (X, \mathcal{F})$ can never be minimal.

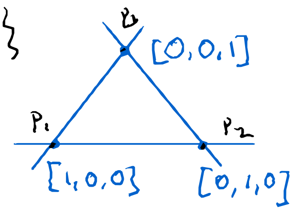
$(X, \mathcal{F})$ is called a non-trivial Riccati foliation.

§ 2.

A very Special foliation.

Let $P_1 = [1,0,0]$, $P_2 = [0,1,0]$, $P_3 = [0,0,1] \in \mathbb{P}^2$, T the automorphism of $\mathbb{P}^2$ with $P_1 \mapsto P_2 \mapsto P_3 \mapsto P_1$. In particular, $T[X_0, X_1, X_2] = [X_2, X_0, X_1]$ $\forall [X_0, X_1, X_2] \in \mathbb{P}^2$, $T^3 = id$, $Fix(T) = \{q_1, q_2, q_3\}$ where q_i are of the form $[1, \beta, \beta^2]$ with $\beta^3 = 1$.

Let $z = \frac{X_0}{X_2}$, $w = \frac{X_1}{X_2}$ & $\mathcal{L}$ the T-invariant foliation generated on $(X_2 \neq 0)$ by $z\frac{\partial}{\partial z} + \left(\frac{1+i\sqrt{3}}{2}\right) w \frac{\partial}{\partial w}$



One can check $Sing(\mathcal{L}) = \{P_1, P_2, P_3\}$ & the lines $\langle \overline{P_1, P_2} \rangle$, $\langle \overline{P_2, P_3} \rangle$ & $\langle \overline{P_3, P_1} \rangle$ are $\mathcal{L}$-invariant. In fact, these are the only $\mathcal{L}$-invariant lines.

Note: $\overline{P_i P_j} \cap Sing\, \mathcal{F} = \phi$ $\forall i, j$.

$(v \sim Tv \sim T^2 v)$

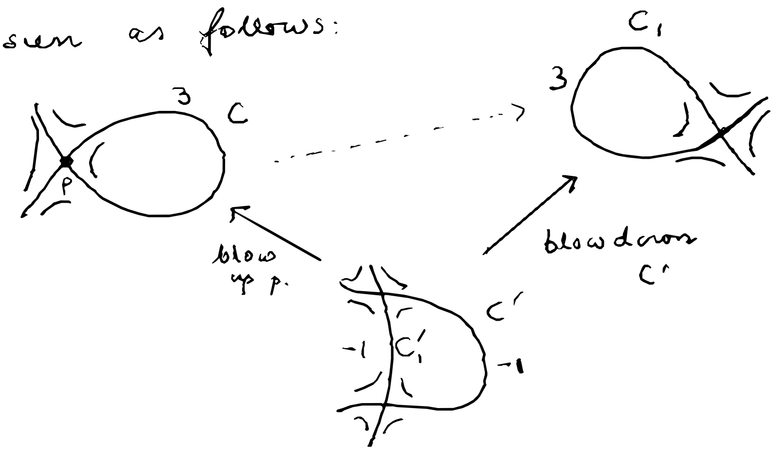
Let $Y_0 = \mathbb{P}^2/T$ with induced foliation $\mathcal{H}_0 = \mathcal{L}/T$.

$Sing(Y_0) = \{\tilde{q}_1, \tilde{q}_2, \tilde{q}_3\}$ coming from $Fix(T)$. Let Y denote the minimal resolution of Y_0 & $\mathcal{H}$ the pullback foliation on it. Over each $\tilde{q}_j$, $j = 1, 2, 3$, Y consists of two (-2) curves D_j, E_j intersecting at a point. In all, $\mathcal{H}$ has 7 foliation invariant curves: (i) $\{D_j, E_j\}_{j=1}^{3}$

(ii) the quotient of $\overline{q_1 q_2} \cup \overline{q_2 q_3} \cup \overline{q_3 q_1}$ on Y = a nodal rational curve C with node p & $C^2 = 3$. p is a reduced non deg sing of $\mathcal{H}$.

[$(Y, \mathcal{H})$ is called a very special foliation]

In particular, there aren't any $\mathcal{H}$-exc curves. $\mathcal{H}$ has a nontrivial birational self map. This can be seen as follows:



Prop (Birational characterisn of Very Special foliation)
Let $\mathcal{F}$ be a foliation on a surface X, C a rational curve with node p, invariant under $\mathcal{F}$, $C^2 = 3$. Suppose that p is a reduced non deg singularity of $\mathcal{F}$ & it's the unique singularity of $\mathcal{F}$ on C. Then $\mathcal{F}$ is birational to $\mathcal{H}$.

Pf: Skipped for now. ∎

Back to Chapter 5:

Example 3: Let $(Y, \mathcal{H})$ be the very special foliation. As noted above there aren't any $\mathcal{H}$-exc curves. Thus $(Y, \mathcal{H})$ is relatively minimal. Since it has a nontrivial birational self map which isn't an isom, $(Y, \mathcal{H})$ isn't minimal.

§3.

Theorem: Let $(X, \mathcal{F})$ be a foliation without a minimal model. Then it is birational to one of the following:

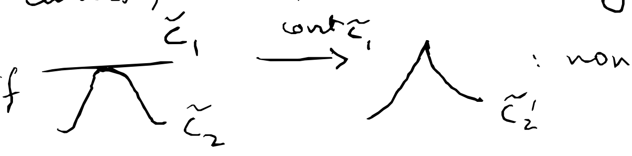
1) a rational fibration,
2) a nontrivial Riccati foliation, or
3) the very special foliation $(Y, \mathcal{H})$.

Pf: Replace $(X, \mathcal{F})$ by a relatively minimal model. Since this is not minimal by assumption, $\exists\ f: (Y, \mathcal{G}) \dashrightarrow (X, \mathcal{F})$ a birational map, which isn't a morphism where $(Y, \mathcal{G})$ has reduced sing. Let

$$\begin{tikzcd} (\tilde{Y}, \tilde{\mathcal{G}}) \arrow[d, "\pi"'] \arrow[dr, "\tilde{f}"] & \\ (Y, \mathcal{G}) \arrow[r, dashed, "f"'] & (X, \mathcal{F}) \end{tikzcd}$$

be a resolution of indeterminacy of f s.t $\tilde{Y}$ is smooth. Then $\tilde{\mathcal{G}}$ has reduced singularities & $\exists\ C \simeq \mathbb{P}^1 \subset \tilde{Y}$ s.t $C^2 = -1$, $\pi(C) = pt$, $\tilde{f}(C) \neq pt$. Thus C is $\tilde{\mathcal{G}}$-exc. Note that $\tilde{f}$ isn't an isom in a nbd of C: ow $\tilde{f}(C)$ would be $\mathcal{F}$-exc. In particular, C intersects some $\tilde{f}$-exc curve; this curve is also $\tilde{\mathcal{G}}$-exc.

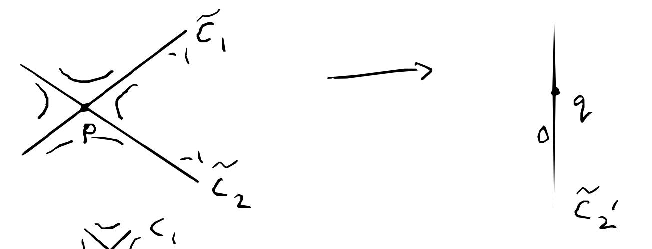
Conclusion: if $(X, \mathcal{F})$ has no minimal model, it is birational to some reduced foliation $(\tilde{X}, \tilde{\mathcal{F}})$ & $\exists\ \tilde{C}_1, \tilde{C}_2 \subset \tilde{X}$ which are $\tilde{\mathcal{F}}$-exc s.t $\tilde{C}_1 \cap \tilde{C}_2 \neq \phi$.

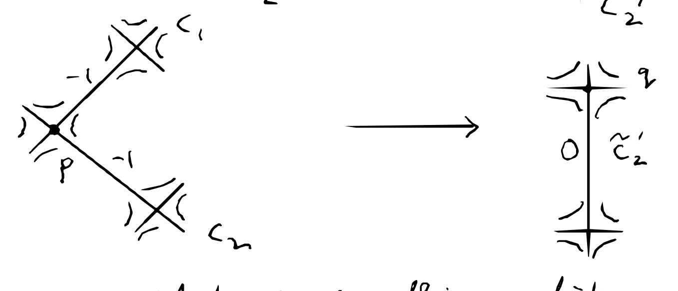
If $p \in \tilde{C}_1 \cap \tilde{C}_2$, then p is reduced nondeg (b/c $Z(\tilde{\mathcal{F}}, \tilde{C}_1, p) = Z(\tilde{\mathcal{F}}, \tilde{C}_2, p) = 1$ by description of $\tilde{\mathcal{F}}$-exceptional curves). Also, $\tilde{C}_1$ transversely intersects $\tilde{C}_2$ (ow if  : non-reduced)

& $\#(\tilde{C}_1 \cap \tilde{C}_2) \leq 2$.

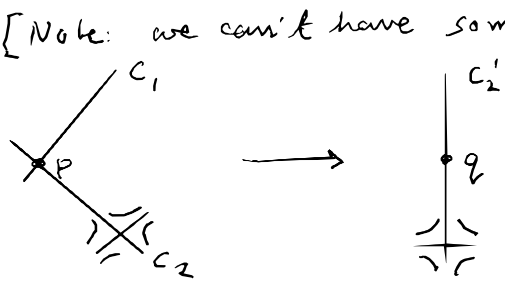
Case a): $\#(\tilde{C}_1 \cap \tilde{C}_2) = 1$.

By contracting $\tilde{C}_1$, $\tilde{C}_2$ transforms into a smooth rational curve $\tilde{C}_2'$ with $\tilde{C}_2'^2 = 0$, $\tilde{C}_2'$ invariant under $\tilde{\mathcal{F}}'$. We have the following possibilities;

① 

② 

[Note: we can't have something like



: indeed $Z(\tilde{\mathcal{F}}', C_2') = 1$ is ruled out by a propn in §1]

If ① occurs, $q \in C_2'$ is regular for $\tilde{\mathcal{F}}'$ & $Z(\tilde{\mathcal{F}}', C_2') = 0$. Then §1 Prop $\Rightarrow$ $\tilde{\mathcal{F}}'$ is a rational fibration with fiber C_2'.

If ② occurs, $Z(\tilde{\mathcal{F}}', C_2') = 2$ & Propn $\Rightarrow$ $\exists$ a rational fibn (with special fiber C_2') st $\tilde{\mathcal{F}}'$ is Riccati wrt it. (That it's nontrivial Riccati is clear from the picture.)

Case b): $\#(C_1 \cap C_2) = 2$. In this case, by contracting C_1, C_2 becomes a nodal rational curve C_2' & one can check $C_2'^2 = 3$. Then $(\tilde{X}', \tilde{\mathcal{F}}')$ is birational to the very special foliation by Propn in §2. ∎