Miyaoka rationality theorem

(Foliations reading seminar)
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Frobenius

X scheme of char $p>0$ ($p=0$ in $\mathcal{O}_X$)

Absolute morphism

$F_X : X \to X$ identity on topological space

$F_X^\# : \mathcal{O}_X \to \mathcal{O}_X \quad f \mapsto f^p$

$(F_X, F_X^\#)$ is a morphism of schemes

Lemma

$f: X \to S$ morphism of schemes in char p

Then

$$\begin{tikzcd} X \arrow[r, "F_X"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & X \arrow[d, "f"] \\ S \arrow[r, "F_S"] & S \end{tikzcd}$$

Note: F_X is not a morphism of S-schemes (unless $F_S = id$)

Relative Frobenius

S = scheme in char $p>0 \qquad f: X \to S \qquad$ (X in char $p>0$)

$X^{(1)} \doteq X \times_{S, F_S} S$

$$\begin{tikzcd} X \arrow[r, dashed, "F"] \arrow[rr, bend left, "\cancel{F_X}"] \arrow[dr, bend right, "f"'] & X^{(1)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ & S \arrow[r, "F_S"] & S \end{tikzcd}$$

$F = F_{X/S}$ relative Frobenius

F is S-linear.

If $X = V(g_1, \dots, g_r) \Rightarrow X^{(1)} = V(g_1^{(p)}, \dots, g_r^{(p)})$

$g_i = \sum \alpha_I X^I \qquad g_i^{(p)} = \sum \alpha_I^p X^I \qquad I$ multiindex

If $(x_1, \dots, x_n) \in X \Rightarrow F(x_1, \dots, x_n) = (x_1^p, \dots, x^p) \in X^{(1)}$

If $X \to S$ is smooth proper, F is finite flat of degree $p^{\dim X}$.

The rationality criterion

X smooth compact complex algebraic surface.
$\mathcal{F}$ foliation on X. If $\exists$ ample divisor H st
$$T_{\mathcal{F}} \cdot H > 0$$
then $\mathcal{F}$ is a foliation by rational curves,
i.e. $\forall x \in X$ $\exists$ rational curve through x and tangent to $\mathcal{F}$.
Hence, up to blowing-up, $\mathcal{F}$ is a rational fibration.
It follows X is uniruled ($\mathbb{P}^1 \times W \dashrightarrow X$ dominant, $\dim W = \dim X - 1$)

Oss 1
No need to blow-up if singularities are non dicritical (e.g. reduced).
If $\mathcal{F}$ is a rational fibration $\exists$ H ample st
$$T_{\mathcal{F}} \cdot H > 0$$

Dual statement
If $\mathcal{F}$ is not a foliation by rational curves, then
$T_{\mathcal{F}}^{\vee}$ is pseudo-effective ($T_{\mathcal{F}}^{\vee} \cdot H \geq 0$ $\forall$ H ample)

Foliations in positive characteristic

X smooth surface $/k$ char $k = p > 0$

A foliation $\mathcal{F}$ on X is given by an open cover $\{U_i\}$, on each U_i a regular vector field v_i with isolated zeros such that on $U_i \cap U_j$
$$v_i = g_{ij} v_j \qquad g_{ij} \in \mathcal{O}_X^*(U_i \cap U_j)$$
and $\{g_{ij}\}$ defines a 1-cocycle in $H^1(X, \mathcal{O}_X^*)$

$\{g_{ij}^{-1}\}$ defines $T_{\mathcal{F}}$ and

$$0 \to T_{\mathcal{F}} \to T_X \to I_Z \cdot N_{\mathcal{F}} \to 0$$

$\dim Z = 0$

Dual sequence

$$0 \to N_{\mathcal{F}}^{\vee} \to \Omega_X^1 \to I_Z \cdot T_{\mathcal{F}}^{\vee} \to 0$$

<u>Ossrv</u>

Let v be a regular vector field on open $U \subset X$

$\Rightarrow$ v is a derivation of $\mathcal{O}_X(U)$

$v : \mathcal{O}_X(U) \to \mathcal{O}_X(U)$ K-linear

$v(fg) = f\,v(g) + v(f)\,g$ (Leibniz)

$n > 0$ integer $v^n = v \circ \dots \circ v$ does not satisfy Leibniz unless $n = p$

$$v^p(fg) = \sum_{k=0}^{p} \binom{p}{k} v^k(f)\, v^{p-k}(g) = v^p(f)\, g + f\, v^p(g)$$

<u>Def</u>

$\mathcal{F}$ is p-closed if: $v \in \mathcal{F} \Rightarrow v^p \in \mathcal{F}$ $\quad (\mathcal{F}^p \subset \mathcal{F})$

$$\mathrm{Ann}(\mathcal{F}) \doteq \{ f \in \mathcal{O}_X \mid v(f) = 0 \quad \forall\, v \in \mathcal{F} \}$$

<u>Note</u>

1) $\mathcal{O}_X^p \subset \mathrm{Ann}(\mathcal{F})$

$$v(f^p) = p f^{p-1} v(f) = 0 \quad \forall\, v \in T_X$$

2) $\mathrm{Ann}(\mathcal{F})$ is an $\overset{\mathcal{O}_X^p}{\cong}\mathcal{O}_{X^{(1)}}$-subalgebra of $\mathcal{O}_X$ (by Leibniz)

$$v(a^p f) = \cancel{p a^{p-1} v(a) f} + a^p v(f) = a^p v(f) = 0 \quad \text{for } f \in \mathrm{Ann}(\mathcal{F})$$

3) $\mathrm{Ann}(\mathcal{F})$ is integrally closed in $\mathcal{O}_X$

Hence $Y = \mathrm{spec}(\mathrm{Ann}(\mathcal{F})) = X/\mathcal{F}$ is a normal surface such that

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[rr, bend right, "F"'] & Y \arrow[r, "\pi'"] & X^{(1)} \end{tikzcd} \qquad (*)$$

Moreover π, π' are purely inseparable and $\deg \pi = \deg \pi' = p$

Conversely, given a diagram $(*)$, can define a foliation $\mathcal{F}$ on X as

$$\mathcal{F} = \mathrm{Ann}(\mathcal{O}_Y) = \{ v \in T_X \mid v(f) = 0 \quad \forall f \in \mathcal{O}_Y \} = \mathrm{Ker}(d\pi)$$

$\mathcal{F}$ is p-closed: $v^p(f) = v^{p-1}(v(f)) = 0 \quad \forall f \in \mathcal{O}_Y$

This is a 1-1 correspondence → <u>Lemma pagina dopo</u>

<u>Locally</u> Given $\mathcal{F}$ p-closed on X: have $\begin{tikzcd} X \arrow[r, "\pi"] \arrow[rr, bend right, "F"'] & Y \arrow[r, "\pi'"] & X^{(1)} \end{tikzcd}$

(z,w) suitable local coordinates on X

(x,y) suitable local coordinates on Y

$(x,y) = \pi(z,w) = (z^p, w)$ and $\mathcal{F}$ is generated by $\frac{\partial}{\partial z}$

$\pi'(x,y) = (x, y^p)$

On Y define foliation $\mathcal{G} = \mathrm{Ker}(d\pi') = \mathrm{im}(d\pi)$

($\mathcal{G}$ is p-closed)

$\mathcal{G}$ is generated by $\frac{\partial}{\partial y}$

On $X^{(1)}$ have foliation $\mathcal{H}$ generated by $\frac{\partial}{\partial z^p}$ $\quad (z^p, w^p)$

Lemma

$$\pi^* K_Y = T_f^{\vee\, op} \otimes N_f^\vee$$

pf

$0 \to T_g \to T_Y \to I_Z \cdot N_g \to 0 \quad \Rightarrow \quad K_Y = T_g^\vee \otimes N_g^\vee$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & T_f \arrow[r, phantom, "="] \arrow[d] & T_f \arrow[d] & \boxed{I_W \cdot N_f} \arrow[d] & \\
0 \arrow[r] & T_f \arrow[r] \arrow[d, "0"] & T_X \arrow[r] \arrow[d] & I_W \cdot N_f \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & \pi^* T_g \arrow[r] \arrow[d] & \pi^* T_Y \arrow[r] \arrow[d] & I_Z \cdot N_g \arrow[r] \arrow[d] & 0 \\
 & \boxed{\pi^* T_g} \arrow[d] & I_Z \cdot N_g \arrow[r, phantom, "="] \arrow[d] & I_Z \cdot N_g \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

$\pi^* d_Y = d_W$

$\Rightarrow \quad \pi^* T_g = I_W \cdot N_f \quad \Rightarrow \quad \pi^* T_g^\vee = N_f^\vee$

Similarly $\pi'^* T_H^\vee = N_g^\vee$

$F^* T_H^\vee = T_f^{\vee\, op}$ by definition

$\parallel$

$\pi^* \pi'^* T_H^\vee = \pi^* N_g^\vee$

$\Rightarrow \pi^* K_Y = \pi^* T_g^\vee \otimes \pi^* N_g^\vee = N_f^\vee \otimes T_f^{\vee\, op}$ $\square$

Lemma

For $p \gg 0$ $\mathcal{F}$ is p-closed after reduction mod p of X.

Pf

$\mathcal{F}$ is given by

$$v_i = g_{ij} v_j \qquad \text{and } \{g_{ij}\} \text{ represents } T_{\mathcal{F}}^{\vee}$$

$\mathcal{F}$ is also given by 1-forms (local regular)

$$\omega_i = f_{ij}\, \omega_j \qquad \text{wt } \{f_{ij}\} \text{ representing } N_{\mathcal{F}}$$

$$\text{and } \omega_j(v_j) = 0 \quad \forall j$$

After reducing mod p:

$$v_i^p = (g_{ij} v_j)^p = g_{ij}^p v_j^p + g_{ij} v_j^{p-1} (g_{ij}^{p-1}) v_j$$

and therefore

$$\omega_i(v_i^p) = f_{ij}\, \omega_j \left(g_{ij}^p v_j^p - g_{ij} v_j^{p-1} (g_{ij}^{p-1}) v_j \right)$$

$$= f_{ij}\, \omega_j (g_{ij}^p v_j^p) = f_{ij}\, g_{ij}^p\, \omega_j(v_j^p)$$

The functions $h_j = \omega_j(v_j^p)$ defines a section of $T_{\mathcal{F}}^{\vee \otimes p} \otimes N_{\mathcal{F}} =: L$

For $p \gg 0$ we have then $(L \cdot H) < 0$

because $$(L \cdot H) = p \underbrace{(T_{\mathcal{F}}^{\vee} \cdot H)}_{<0 \text{ by hp.}} + (N_{\mathcal{F}} \cdot H) < 0$$

$\Rightarrow L$ has no global sections $\neq 0$

$$\Rightarrow h_j = \omega_j(v_j^p) = 0 \quad \forall j \quad \Rightarrow \quad v_j^p \in \mathcal{F} \quad \forall j$$

$$(\omega_j(v) = 0 \Leftrightarrow v \in \mathcal{F}) \qquad \square$$

<u>Proof Theorem</u>

By the lemma for $p \gg 0$ can consider $Y = X/\mathcal{F}$

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[rr, bend right, "F"'] & Y \arrow[r, "\pi'"] & X^{(1)} \end{tikzcd}$$

May suppose H very ample.

Let $x \in X$ general point and $x \in C \subset X$, $C \sim H$ an irreducible curve not tangent to $\mathcal{F} = \ker(d\pi)$

Then $\pi|_C : C \to C' = \pi(C)$ is birational because it is separable and bijective

$$K_Y \cdot C' = \pi^* K_Y \cdot C = p(T_{\mathcal{F}}^\vee \cdot C) + N_{\mathcal{F}}^\vee \cdot C < 0$$

for $p \gg 0$

<u>Mori bend and break</u>

Let $H^{(1)}$ v.a. on $X^{(1)}$ st $F^* H^{(1)} = H^{\otimes p}$ (let $H' = \pi'^* H^{(1)}$)

Then $\exists$ rational curve $R' \subset Y$ through $x' = \pi(x)$ and

$$R' \cdot H' \leq \frac{2(H' \cdot C')}{-(K_Y \cdot C')} = \frac{2p(H \cdot C)}{p\, T_{\mathcal{F}} \cdot C + N_{\mathcal{F}} \cdot C}$$

$$= \frac{2(H \cdot C)}{T_{\mathcal{F}} \cdot C + \frac{1}{p}(N_{\mathcal{F}} \cdot C)} \leq K$$

where K costant independent of p and x.

Let $R = \pi^{-1}(R') \subset X$. R is a rational curve because π bijective. Note $x \in R$.

<u>R is tangent to $\mathcal{F}$.</u>

<u>Pf</u>: If not, $\pi : R \to R'$ is birational and $R' \cdot H' = R \cdot \pi^*(H') = p\,(R \cdot H) \leq K$

$$\Rightarrow R \cdot H \leq \frac{1}{p} K < 1$$

for $p \gg 0$ ↯

<u>Now prove $R \cdot H \leq K$</u>

<u>Pf</u>: As $\pi : R \to R'$ is purely inseparable of degree p

$$\Rightarrow p(R \cdot H) = R \cdot \pi^*(H') = p\,(R' \cdot H') \leq p\,K \quad \square$$

To conclude, through general $x \in X$, we found a rational curve on X mod p, tangent to $\mathcal{F}$ and whose H-degree is uniformly bounded.
All of this enables us to lift the rational curve to char 0 keeping the tangency and degree.
Can extend this open family of rational curves to a rational fibration on X (with possibly some indeterminancy point) tangent to $\mathcal{F}$.