Recall: ① Let $C = V(f)$ is an $\mathcal{F}$-invariant curve, $\mathcal{F} \leadsto \omega$ (around $p \in C$). We can write $g\omega = h\,df + f\eta$, so $\omega|_C = \frac{h}{g}\big|_C \cdot df$. $Z(\mathcal{F}, C, p) \doteq$ vanishing # of $\frac{h}{g}\big|_C$ at p [If $p \in C_{sm}$, this is also vanishing # of $v|_C$ at p].

$$Z(\mathcal{F}, C) \doteq \sum_{p \in C} Z(\mathcal{F}, C, p).$$

② If $C = V(f)$ is non-$\mathcal{F}$-invariant, $\mathcal{F} = \mathcal{O}\langle v\rangle$ around $p \in C$. Then $\mathrm{tang}(\mathcal{F}, C, p) \doteq \dim_{\mathbb{C}} \frac{\mathcal{O}_{X,p}}{\langle f, v(f)\rangle}$ measures the tangency # of $\mathcal{F}$ & C at p.

$$\mathrm{tang}(\mathcal{F}, C) \doteq \sum_{p \in C} \mathrm{tang}(\mathcal{F}, C, p).$$

Prop1: If C non-$\mathcal{F}$-invariant, then $N_{\mathcal{F}} \cdot C = \chi(C) + \mathrm{tang}(\mathcal{F}, C)$ & $T_{\mathcal{F}} \cdot C = C^2 - \mathrm{tang}(\mathcal{F}, C)$.

Prop2: If C is $\mathcal{F}$-invariant, then

$$N_{\mathcal{F}} \cdot C = C^2 + Z(\mathcal{F}, C), \quad T_{\mathcal{F}} \cdot C = \chi(C) - Z(\mathcal{F}, C).$$

Def: Let $\pi : X \to B$ be a rational fibration. A foliation $\mathcal{F}$ on X is Riccati wrt π if it is transverse to a general fiber of π.

Prop: Let $\mathcal{F}$ be a foliation on X & $F \simeq \mathbb{P}^1 \subset X$ s.t $F^2 = 0$ & F is $\mathcal{F}$-invariant. $\exists$ a rational fibration $\pi : X \to B$ with fiber F. (standard fact). We then have:

① If $Z(\mathcal{F}, F) = 0$, $\mathcal{F} = T_{X/B}$.

② If $Z(\mathcal{F}, F) = 2$, $\mathcal{F}$ is Riccati wrt π.

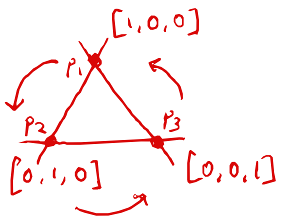
③ $Z(\mathcal{F}, F) \neq 1$.

Pf: Let $Z(\mathcal{F}, F) = 0$, $F' \neq F$ another fiber of π. S'pose F' isn't $\mathcal{F}$-invariant. Prop1: $T_{\mathcal{F}} \cdot F' = -\mathrm{tang}(\mathcal{F}, F')$, Prop2: $T_{\mathcal{F}} \cdot F = 2$. This gives $\mathrm{tang}(\mathcal{F}, F') = -2$, which is absurd $\therefore$ Every fiber of π is $\mathcal{F}$-invariant $\therefore \mathcal{F} = T_{X/B}$ ($\because$ two foliations agreeing on a dense open are the same)

Let $Z(\mathcal{F}, F) = 2$. $Z(\mathcal{F}, F, p) > 0$ for some $p \in F$. If $\mathcal{F} \leadsto \omega$, this means $\omega|_C$ vanishes at p. In particular, $p \in \mathrm{Sing}\,\mathcal{F}$. Let F' be a general fiber of π & assume it's $\mathcal{F}$-invariant. Then $Z(\mathcal{F}, F') = 2$ [$\because Z(\mathcal{F}, C) = \chi(C) - T_{\mathcal{F}} \cdot C$], giving $F' \cap \mathrm{Sing}\,\mathcal{F} \neq \emptyset$, which is absurd $\therefore F' \pitchfork \mathcal{F}$ & $\mathcal{F}$ is Riccati wrt π.

Proof of ③ is similar. ∎

"Very Special foliation" [Topic for 11th; Skipped]



$T \curvearrowright \mathbb{P}^2$

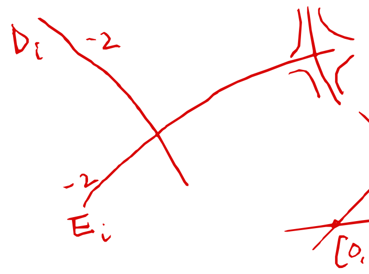
$T([x_0, x_1, x_2]) = [x_2, x_0, x_1]$.

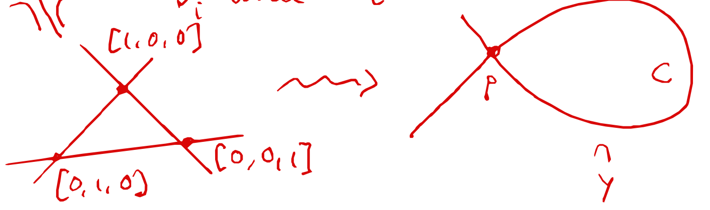
$T^3 = id$. Fix $T = \{q_1, q_2, q_3\}$

(q_i of the form $[1, \beta, \beta^2]$, where $\beta^3 = 1$).

Let $z = \dfrac{x_0}{x_3}$, $w = \dfrac{x_2}{x_3}$. One can check that the foliation $\mathcal{L}$ generated in $(x_3 \neq 0)$ by the vector field(s) $v = z \cdot \dfrac{\partial}{\partial z} + \dfrac{1}{2}(1 \pm i\sqrt{3})\, w \dfrac{\partial}{\partial w}$ is (are) T-invariant. Let $\mathcal{H}_0$ be the induced foliation on $Y_0 = \mathbb{P}^2/T$. The fixed pts q_1, q_2, q_3 of T give the three singular pts of Y_0, say $\tilde{q}_1, \tilde{q}_2, \tilde{q}_3$. Let $Y \to Y_0$ be the minimal resoln & $\mathcal{H}$ the induced foliation on Y. Over each $\tilde{q}_i$, Y consists of two -2 curves D_i & E_i.

Since q_i aren't sing pts of $\mathcal{L}$, D_i and E_i are $\mathcal{H}$-invariant.

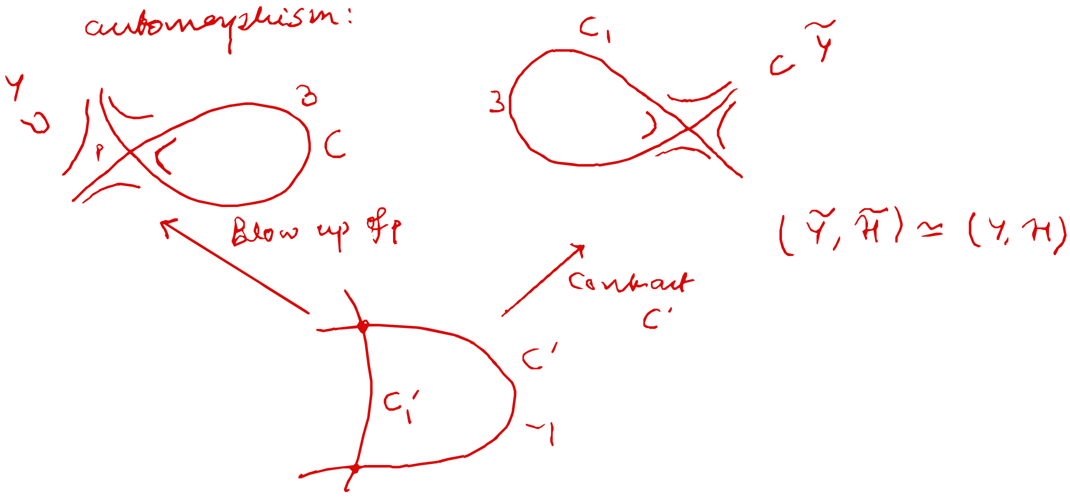




$C^2 = 3$, C is $\mathcal{H}$-invariant.

$(D_i, E_i)_{i=1,2,3}$ & C are the only $\mathcal{H}$-invariant curves.

Claim: $\mathcal{H}$ has a non-trivial birational automorphism:



This observation has the following partial converse (proof skipped for now).

Prop: Let F be a foliation on X, $C \subset X$ an F-invariant rational curve with node p with $C^2 = 3$. Suppose p is a reduced nondeg sing of F & that it's the unique singularity of F on C. Then F is birational to $\mathcal{H}$.

Def Given a foliation $(X, \mathcal{F})$, a curve $C \subset X$ is called $\mathcal{F}$-exceptional if:

① $C \simeq \mathbb{P}^1$ & $C^2 = -1$;

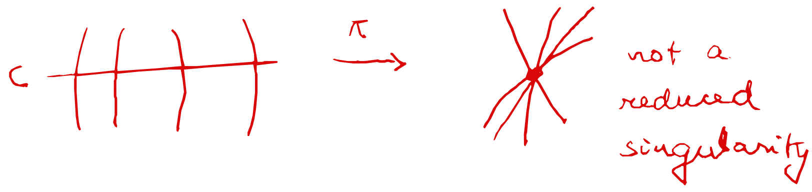
② If $\pi : X \to X'$ is the blow-down of C, then

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \cup & & \cup \\ C & \longmapsto & p \end{array}$$

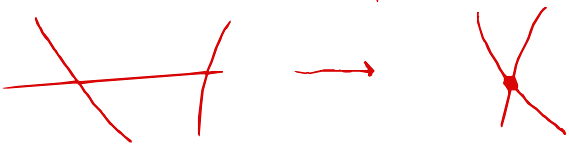
$\mathcal{F}'$ has at p either a regular pt, or a reduced singular pt.

Remarks:

a) An $\mathcal{F}$-exc curve C can't be non $\mathcal{F}$-invariant:



& it can contain max of 2 sing points of $\mathcal{F}$:



b) Recall (Saverio's talk): Let $\mathcal{F}$ be a foliation on X, $\pi : \tilde{X} \to X$ the blowup of p. Let $\mathcal{F} \leadsto \omega$,

$$\begin{array}{ccc} \tilde{X} & \to & X \\ \cup & & \cup \\ E & \longmapsto & p \end{array}$$

$a(p) =$ vanishing order of ω at p,

$l(p) =$ vanishing order of $\pi^*\omega$ along E. Then

$$l(p) = \begin{cases} a(p) & \text{if } E \text{ is } \tilde{\mathcal{F}}\text{-invariant (good case)} \\ a(p)+1 & \text{if } E \text{ is non } \tilde{\mathcal{F}}\text{-invariant (bad case)} \end{cases}$$

$$N^*_{\tilde{\mathcal{F}}} = \pi^* N^*_{\mathcal{F}} \otimes \mathcal{O}_{\tilde{X}}(l(p)E) \quad \therefore \quad N_{\tilde{\mathcal{F}}} \cdot E = l(p).$$

If E is $\tilde{\mathcal{F}}$-invariant (which is our case), then

$$\text{Prop 2} \Rightarrow Z(\tilde{\mathcal{F}}, E) = N_{\tilde{\mathcal{F}}} \cdot E - E^2 = l(p) + 1 = a(p) + 1.$$

Now let $\pi : \begin{array}{ccc} C & \longmapsto & p \\ \cap & & \cap \\ X & \to & X' \end{array}$ be the contraction of an $\mathcal{F}$-exc curve C.

$$a(p) = \begin{cases} 0 & \text{if } p \in \mathcal{F}'_{sm} \\ 1 & \text{if } p \text{ is a reduced sing} \end{cases}$$

$$\therefore Z(\mathcal{F}, C) = \begin{cases} 1 & \text{if } p \in \mathcal{F}'_{sm} \ (*) \\ 2 & \text{if } p \text{ is a reduced sing} \ (\#) \end{cases}$$

With this, observe that $\mathcal{F}$-exc curves C can be of the following two kinds:

① C contains only one singularity q of $\mathcal{F}$ with $Z(\mathcal{F}, C, q) = 1$. (This corresponds to $(*)$). Then $p = \pi(C) \in \mathcal{F}'_{sm}$.

② C contains two sing pts q_1, q_2 of $\mathcal{F}$ with $Z(\mathcal{F}, C, q_1) = Z(\mathcal{F}, C, q_2) = 1$ (This corresponds to $(\#)$). Then $p = \pi(C)$ is a reduced sing of $\mathcal{F}'$.

[Note: $Z(\mathcal{F}, C, q) = 2$ can't happen].

def: $(X, \mathcal{F})$ is called <u>relatively minimal</u> if $\mathcal{F}$ has reduced singularities & X contains no $\mathcal{F}$-exc curves.

Prop: Any $(X,\mathcal{F})$ has a relatively minimal model.

Pf: Seidenberg: $\exists\ (\tilde{X}, \tilde{\mathcal{F}}) \longrightarrow (X,\mathcal{F})$ s.t $\tilde{\mathcal{F}}$ has only reduced sings. We keep on blowing down $\tilde{\mathcal{F}}$-exc curves. Due to drop of Picard #, this eventually stops. [Blow down of $\tilde{\mathcal{F}}$-exc curves preserves reduced sings, by def].

Remark: Relatively minimal model may not be unique.

def: $(X,\mathcal{F})$ is called <u>minimal</u> if:
① It's relatively minimal &
② If $(Y,\mathcal{G})$ is relatively minimal & biratl to $(X,\mathcal{F})$, then it is isom to $(X,\mathcal{F})$.

Note: Taking $(Y,\mathcal{G}) = (X,\mathcal{F}) \Rightarrow$ any biratl self-map of $(X,\mathcal{F})$ is an automorphism.

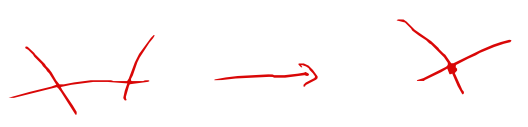
Ex 1: Let $\pi: X \longrightarrow B$ be a rational fibn & $\mathcal{F} = T_{X/B}$. Then

Claim 1: $\mathcal{F}$ is always reduced.

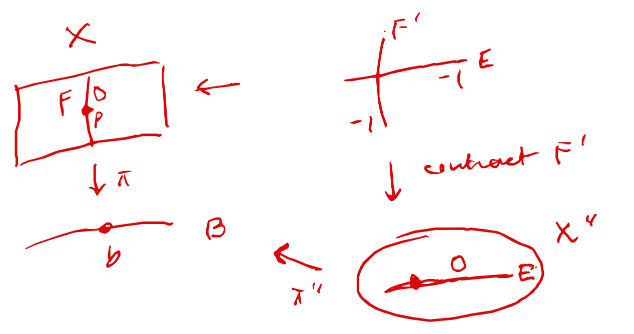
Claim 2: $\mathcal{F}$ is relatively minimal iff all fibers of π are smooth

Claim 3: $\mathcal{F}$ is never minimal.

Pf: X is obtained by blowing up a $\mathbb{P}^1$-bundle over B. The singular fibers of π consist of a tree of $\mathbb{P}^1$'s, each component being a (-1) curve. $\therefore$ $\mathcal{F}$ is reduced. ② is also clear from this.

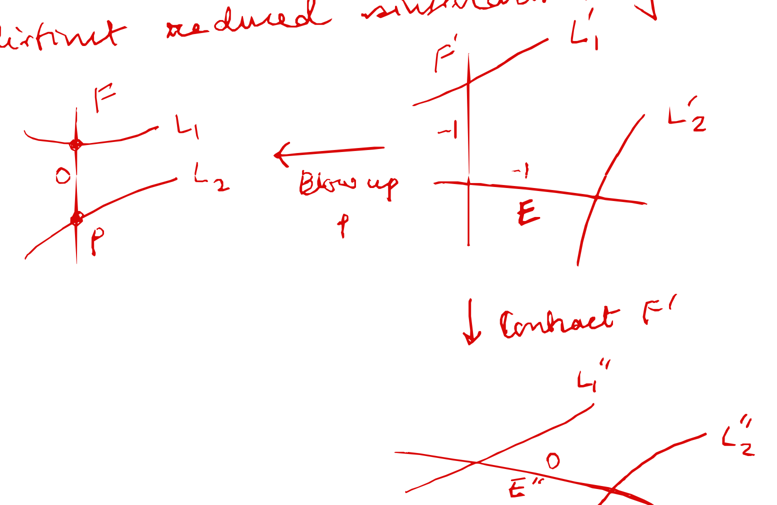


(†) See footnote below



If X is rel minimal, so is X'', but X & X'' are never isomorphic.

Ex 2: Let $\mathcal{F}$ be Riccati wrt $\pi: X \to B$ a $\mathbb{P}^1$-fibration. Let F be a smooth fiber of π which is $\mathcal{F}$-invariant s.t F contains two distinct reduced singularities of $\mathcal{F}$.



Note: If X is rel minimal, so is X''. But X & X'' can't be isomorphic. But X'' (& hence X also) can't be minimal. □

Prop: Let $(X,\mathcal{F})$ be a reduced foliation. Then
① $(X,\mathcal{F})$ is relatively minimal iff any birational morphism $(X,\mathcal{F}) \xrightarrow{\pi} (Y,\mathcal{G})$ onto a reduced foliation is an isom.

Pf: Easy b/c if π is not an isom, then it's a seq of blowups.

② $(X,\mathcal{F})$ is minimal iff any bimeromorphic map $(Y,\mathcal{G}) \dashrightarrow (X,\mathcal{F})$ from a reduced foliation is a morphism.

Pf: Suppose $(X,\mathcal{F})$ is minimal. Consider

$$\begin{tikzcd} & (Y_0,\mathcal{G}_0) \arrow[dr, dashed, "h"] \arrow[d, phantom, "\circlearrowleft"] & \\ (Y,\mathcal{G}) \arrow[ur, "g"] \arrow[rr, dashed, "f"'] & & (X,\mathcal{F}) \end{tikzcd}$$

where $(Y_0,\mathcal{G}_0)$ is a relatively minimal model of $(Y,\mathcal{G})$. Then h is an isom.

$\therefore$ $f = h \circ g$ is a morphism.

Suppose $(X,\mathcal{F})$ satisfies the stated property. If $C \subset X$ is $\mathcal{F}$-exc, contracting C gives $(X,\mathcal{F}) \longrightarrow (X',\mathcal{F}')$ whose inverse is not a morphism $\therefore$ $(X,\mathcal{F})$ is relatively minimal.

If $(Y,\mathcal{G})$ is relatively minimal & have $(Y,\mathcal{G}) \overset{f}{\dashrightarrow} (X,\mathcal{F})$ birational, then f is a morphism by assumption.

Now $(X,\mathcal{F}) \overset{f^{-1}}{\dashrightarrow} (Y,\mathcal{G})$ is a morphism by ①. ∎

(†) <u>Footnote</u>: If $X = \mathbb{P}\mathcal{E} \xrightarrow{\pi} B$ & $X'' = \mathbb{P}\mathcal{E}'' \xrightarrow{\pi''} B$, $\mathcal{E}''$ can be described explicitly in terms of $\mathcal{E}$. See Beauville's "Complex Alg Surfaces" Exc III.24(2)