Incontro 3 | $\boxed{\text{GDS - Foliazioni}}$ | 13/11/23 | 1

$\boxed{\text{Ricordiamo}}$ X = Varietà normale.

Una foliazione $\mathcal{F}$ è un sottofascio $T_{\mathcal{F}} \subset T_X$ del fascio tangente, t.c.

(1) $T_{\mathcal{F}}$ is saturated, i.e. $T_X/T_{\mathcal{F}}$ is torsion free.

(2) $[T_{\mathcal{F}}, T_{\mathcal{F}}] \subseteq T_{\mathcal{F}}$.

Consequences:

(1) $T_{\mathcal{F}}$ is reflexive. $\Rightarrow$ If X = smooth, then $T_{\mathcal{F}}$ is locally free in codimension 2.

(2) $T_X/T_{\mathcal{F}}$ is torsion free $\Rightarrow$ $\mathrm{Sing}(\mathcal{F})$ has codimension 2.

$\boxed{X \text{ smooth surface}}$

- $T_{\mathcal{F}}$ is reflexive of rank 1 $\Rightarrow$ $T_{\mathcal{F}}$ is locally free.

This fact is true for any smooth variety: reflexive of rank = 1 are locally free

⚠ There is an exact sequence of sheaves

$$0 \longrightarrow T_{\mathcal{F}} \xrightarrow{\varphi} T_X \longrightarrow T_X/T_{\mathcal{F}} \simeq \mathcal{I}_Z \otimes N_{\mathcal{F}} \to 0,$$

where $Z = \mathrm{Sing}(\mathcal{F})$ and $N_{\mathcal{F}}$ is locally free of rank 1.

But $\mathrm{Sing}(\mathcal{F}) := \{x \in X \mid T_{\mathcal{F},x} \text{ is not a subline bundle of } T_{X,x}\}$,

so, $T_{\mathcal{F}} \to T_X$ is not injective and sends

(The induced map on the associated line bundles)

$T_{\mathcal{F},x} \longmapsto 0 \quad \forall x \in \mathrm{Sing}(\mathcal{F})$.

In fact φ gives a map of vector bundles iff $\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\varphi)$ are locally free of constant rank.

Recall $\mathcal{F}$-invariant curve

Let $\mathcal{F}$ be given by V, $0 \to T_{\mathcal{F}} \to T_X$, and let $C \subset X$ be an irr. curve.
$\updownarrow$ V = vector field associated to $\mathcal{F}$.

• C is $\mathcal{F}$-invariant iff $V(f) \in (f)$ in $\mathcal{O}_{X,p}$, $\forall p \in C$.
$\|$ loc. vicino a p
$\{f=0\}$

• Equiv. se V é tangente a C, i.e.

$$\begin{tikzcd} 0 \arrow[r] & T_{\mathcal{F}|_C} \arrow[r] \arrow[dr, dashed, "\exists"'] & T_{X|_C} \\ & & T_C \arrow[u] \end{tikzcd}$$

Commento:
$V(f) = df(V) \in (f)$
e $df(V)_{|C} = 0 \iff V_{|C} \in T_C$
Esempio: $C = \{f=0\}$
$V = A\partial_x + B\partial_x$
Il vettore $(f_x, f_y)_{|C}$ é normale a C
e $V(f) = A\partial_x f + B\partial_y f = 0$
implica $(A,B) \perp (f_x, f_y)$.

Definizione Let $\mathcal{F}$ foliation and $p \in \mathrm{Sing}(\mathcal{F})$.

① A separatrix of $\mathcal{F}$ at p is an irreducible curve C s.t. $p \in C$ and C is $\mathcal{F}$-invariant.

[~~i.e. C is a leaf of $\mathcal{F}$ and $p \in C$~~]

② $p \in \mathrm{Sing}(\mathcal{F})$ is called dicritical iff $\exists$ infinitely many separatrices through p.

Definition Let $\mathcal{F}$ be a foliation and $p \in \mathrm{Sing}(\mathcal{F})$. Let V be a vector field defining $\mathcal{F}$, and let $D(\mathcal{F})$ be its Jacobian.
Let λ_1, λ_2 be the eigenvalues of $D(\mathcal{F})(p)$.

① We say that p is a reduced singularity if
(i) at least one of $\{\lambda_1, \lambda_2\}$ is $\neq 0$
(ii) $\lambda := \frac{\lambda_1}{\lambda_2} \vee \frac{\lambda_2}{\lambda_1} \notin \mathbb{Q}^+$.

Nota • λ and λ^{-1} are interchangeable, and $\lambda :=$ eigenvalue of $\mathcal{F}$ in p.

• h = hol. nowhere vanishing $\Rightarrow$ V and hV give same foliation and $J(V)(p) = J(hV)(p)$. ✓

② A reduced singularity p is called.

- **nondegenerate**, if $\lambda_1, \lambda_2 \neq 0$, i.e. $\lambda \neq 0$;
- **saddle-node**, otherwise.

Remark Reduced sing. differ depending on λ.

(1) (Poincaré domain) $\lambda \notin \mathbb{R}^{\leq 0}$

$\mathcal{F}$ is linearizable around p, i.e. $\exists\, p \in \mathcal{U}$ $\underset{(0,0)}{\shortparallel}$ s.t. $\mathcal{F}$ is given on $\mathcal{U}$ by

$$V = x\partial_x + \lambda y \partial_y \qquad (\longleftrightarrow \omega = x\,dy - \lambda y\,dx).$$

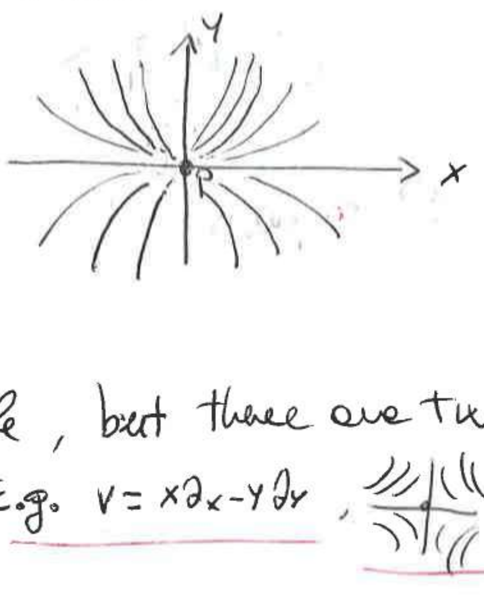
There are two separatrices: $\{x=0\}$ and $\{y=0\}$.

Computation: the integral curves are $\gamma(t) = (a e^{t}, b e^{\lambda t})$,

i.e. $C = \{a^{\lambda} \cdot y = b\, x^{\lambda}\}$.

(E.g. $\lambda \in \mathbb{C} \setminus \mathbb{R}$ or $\lambda \in \mathbb{R}^+ \setminus \mathbb{Q}^+$.)

e.g. $\lambda = \sqrt{2}$



(2) (Siegel domain) $\lambda \in \mathbb{R}^-$

$\mathcal{F}$ is not always linearizable, but there are two separatrices given by $\{x=0\}$ and $\{y=0\}$. E.g. $v = x\partial_x - y\partial_y$,  , $\lambda = -1$,

(3) (Saddle-node) $\lambda = 0$

In suitable coordinates, $p = (0,0) \in \mathcal{U}$ and

$$v = [x(1+\nu y^k) + y \cdot F(x,y)]\partial_x + y^{k+1}\partial_y, \quad \text{with}$$

$k \in \mathbb{N}^+$, $\nu \in \mathbb{C}$ and $F(x,y) =$ hol. with $\mathrm{ord}_p F \leq k$.

$\{y=0\} :=$ strong separatrix always exists. Sometimes $\exists$ another separatrix called "weak separatrix", and transverse to $\{y=0\}$. (E.g. $F \equiv 0$).

Data S = superficie liscia e $\mathcal{F}$ una foliazione su S data localm. da una 1-forma $\omega = A(x,y)\,dx + B(x,y)\,dy$. **3**

Definizione Sia $p \in \mathrm{Sing}(\mathcal{F})$.

Il "vanishing order" di ω in p è dato dal minimo tra $\mathrm{ord}_p A$ e $\mathrm{ord}_p B$, e lo denoto con $\underline{a(p)}$.

Ricordo: data una funzione olomorfa $A(x,y)$ t.c. $A(p)=0$
$\mathrm{ord}_p A$ è il $\min\{k \mid$ il vettore $\left(\frac{\partial A}{\partial x^i \partial y^j}\right), i+j=k$, non si annulla in $p.\}$

Nota: Se $\mathrm{ord}_p A = \alpha \Rightarrow A(x,y) = \sum_{d \geq \alpha} \underbrace{A_d(x,y)}_{\text{polinomi omogenei di grado } d}$.
Supp. $p=(0,0)$

Remark

Sia $\mathcal{F}$ su S data da ω e sia $\pi : \tilde{S} \to S$, $E \mapsto p$ il blow-up lungo $p \in \mathrm{Sing}(\mathcal{F})$.

Si definisce una foliazione $\tilde{\mathcal{F}}$ su $\tilde{S}$ indotta da $\mathcal{F}$:

considero $\pi^*\omega$ e sia $\underline{l(p) := \text{"vanishing order" di } \pi^*\omega \text{ in } E}$.
Sia $f=0$ equazione locale di E.
Allora la 1-forma $\tilde{\omega} = \frac{\pi^*\omega}{f^{l(p)}}$ definisce una foliazione $\tilde{\mathcal{F}}$.

Lemma We have either:

1) $l(p) = a(p)+1$, if E is <u>not</u> $\tilde{\mathcal{F}}$-invariant;

2) $l(p) = a(p)$, if E is $\tilde{\mathcal{F}}$-invariant.

Proof $\tilde{S}$ is locally given by the equation $xt = ys$ over an open set of S with local coord. (x,y).

$\pi^{-1}(\mathcal{U}) \longrightarrow \underset{\mathbb{C}^2}{\overset{\cup}{\mathcal{U}}} \ni (x,y)$ and $\pi^{-1}(\mathcal{U}) = \underset{\mathbb{C}^2}{\overset{\cong}{V_t}} \cup \underset{\mathbb{C}^2}{\overset{\cong}{V_s}}$, wher $V_t = \{t \neq 0\}$
$V_s = \{s \neq 0\}$

Over V_t, $x = sy$ and $\pi^*\omega = y \cdot A(sy,y) \cdot ds + [s \cdot A(sy,y) + B(sy,y)]\, dy$,

so that $\tilde{\omega} = \tilde{A}(s,y)\, ds + \tilde{B}(s,y)\, dy$ is the holom. 1-form defining $\tilde{\mathcal{F}}$,

and $\begin{cases} \tilde{A}(s,y) = y^{1-\ell(P)} \cdot A(sy,y) \\ \tilde{B}(s,y) = y^{-\ell(P)} \cdot [s \cdot A(sy,y) + B(sy,y)] \end{cases}$. Observe that $E = \{y=0\}$.

Suppose E $\tilde{\mathcal{F}}$-invariant:

In general

$\alpha = \mathrm{ord}_P A \Rightarrow A(sy,y) = y^{\alpha} \cdot f(s,y)$, $f(s,0) \neq 0$, f, g holom.

$\beta := \mathrm{ord}_P B \Rightarrow B(sy,y) = y^{\beta} \cdot g(s,y)$, $g(s,0) \neq 0$

$a(P) = \min\{\alpha, \beta\}$

$\Rightarrow \begin{cases} \tilde{A}(s,y) = y^{\alpha+1-\ell(P)} \cdot f(s,y) \\ \tilde{B}(s,y) = y^{-\ell(P)} \cdot [y^{\alpha} s f + y^{\beta} g] \end{cases}$

$\boxed{\text{Fact}}$

(1) Se $\mathrm{ord}_E(y^{\alpha} s f + y^{\beta} g) \geq a(P) + 1 \Rightarrow \alpha = \beta$ e $a(P) + 1 \geq \ell(P)$

(2) Se $\mathrm{ord}_E(y^{\alpha} s f + y^{\beta} g) = a(P) \Rightarrow a(P) \geq \ell(P)$

Se E è $\tilde{\mathcal{F}}$-inv. $\Rightarrow \dfrac{A(sy,y)}{y^{\ell(P)-1}} \in (y) \Rightarrow \alpha + 1 > \ell(P)$.

Se siamo in caso (1) $\Rightarrow a(P) = \alpha = \beta$ e $a(P) + 1 - \ell(P) > 0$

$\Rightarrow \tilde{A}/y$ e $\tilde{B}/y$ sono ancora olomorfe ✗ (Contraddice massimalità di $\ell(P)$)

Allora siamo in caso (2) e $\tilde{\omega} = y^{\alpha+1-\ell(P)} \cdot f\, ds + y^{a(P)-\ell(P)} \cdot [\text{holom. e non si annulla in } E]\, dy$ $\Rightarrow \boxed{a(P) = \ell(P)}$

Se E not $\tilde{\mathcal{F}}$-inv.:

As above, $\alpha + 1 = \ell(P)$.

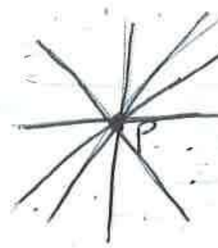
Case (2) $\Rightarrow \alpha \geq a(P) \geq \ell(P) = \alpha + 1$ ✗

Case (1) $\Rightarrow \alpha = a(P) = \beta$ and $\boxed{\ell(P) = a(P) + 1}$

$\boxed{\text{Examples}}$ $S = \mathbb{C}^2$, $\tilde{S} = Bl_p S$.

① $\omega = x\,dy - y\,dx$ $\quad (\longleftrightarrow v = x\partial_x + y\partial_y)$

$C = \{ax = by\}$



The singularity $p = (0,0)$ is dicritical (infinitely many separatrices), it is non-reduced $(\lambda = 1)$, $a(p) = 1$.

Let $\tilde{S} = Bl_p S$:

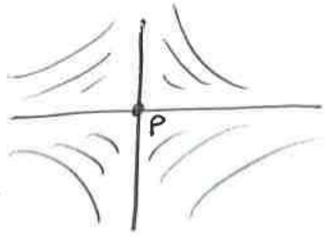
Over $V_t : x = sy$ and $\pi^*\omega = -y^2 ds \implies \ell(p) = 2$.

$\tilde{\omega} = -ds$ $(\longleftrightarrow \tilde{v} = \partial_y)$ and E is not $\tilde{\mathcal{F}}$-inv.

$[\ell = a+1$ or anyway $\tilde{v}(y) = 1 \notin (y)]$

② $\omega = x\,dy + y\,dx$ $\quad (\longleftrightarrow v = x\partial_x - y\partial_y)$ $\qquad C = \{xy = a\}$

The sing. $p = (0,0)$ is nondicritical (two separatrices), it is reduced (Siegel) $(\lambda = -1)$, $a(p) = 1$.



Let $\tilde{S} = Bl_p S$:

Over $V_t : x = sy$ and $\pi^*\omega = 2sy\,dy + y^2 ds \implies \ell(p) = 1$.

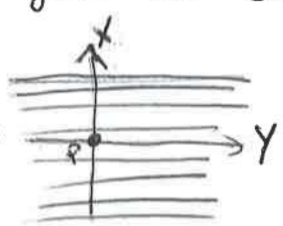
$\tilde{\omega} = 2s\,dy + y\,ds$ $(\longleftrightarrow \tilde{v} = -y\partial_y + 2s\partial_s)$ and E is $\tilde{\mathcal{F}}$-inv.

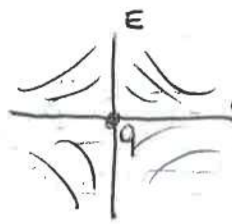
$[\ell = a$ or $\tilde{v}(y) = -y \in (y)]$

③ Blow-up smooth point

$\omega = dx$, $\tilde{\omega} = \pi^*\omega = s\,dy + y\,ds$ $[x = sy]$, $\tilde{\omega} = \pi^*\omega = dx$ $[y = tx]$
$(v = \partial_y)$

$\tilde{\omega}$ gets a singular point $q = (x,y), (s:t) = (0,0), (0:1)$.

$\mathcal{F} =$ 

$\longleftarrow$ $\tilde{\mathcal{F}} =$ 

"Assey" $= \{s = 0\}$

$q \in Sing(\tilde{\mathcal{F}})$ is reduced! (True in general)

Moreover

• E é $\tilde{\mathcal{F}}$-inv.

On $x = sy$: $\tilde{v} = y\partial_y - s\partial_s$ and $\tilde{v}(y) \in y$.

• Ma anche l' "Assey" é $\tilde{\mathcal{F}}$-inv.

(IV) Examples

(1) Blow-up :

Definition $\mathcal{F}$ on X, $\pi : \widetilde{X} \longrightarrow X$ blow up along p.
On $\widetilde{X}$ we can define a foliation $\widetilde{\mathcal{F}}$:

$\mathcal{F} \longleftrightarrow$ hol. 1-form $\omega \rightsquigarrow \omega^* = \pi^*\omega \rightsquigarrow$ extend ω^* to $\widetilde{\omega}$ on $\widetilde{X}$,

so that $\widetilde{\omega}$ has isolated zeros. Define $a(p)$ = vanish. order of ω at p.

Define $\ell(p)$ = vanishing order of $\pi^*\omega$ on E.

Fact • If E is $\widetilde{\mathcal{F}}$-invariant $\Rightarrow \ell(p) = a(p)$;
• If E is not $\widetilde{\mathcal{F}}$-invariant $\Rightarrow \ell(p) = a(p) + 1$.

• $K_{\widetilde{X}} = \pi^* K_X \otimes \mathcal{O}_{\widetilde{X}}(E)$
• $N_{\widetilde{\mathcal{F}}} \simeq \pi^* N_{\mathcal{F}} \otimes \mathcal{O}_{\widetilde{X}}(-\ell(p)E)$ $\Rightarrow T_{\widetilde{\mathcal{F}}} \simeq \pi^* T_{\mathcal{F}} \otimes \mathcal{O}_{\widetilde{X}}((\ell(p)-1)E)$.

Case E is $\widetilde{\mathcal{F}}$-invariant

$$Z(\widetilde{\mathcal{F}}, E) = -E^2 + N_{\widetilde{\mathcal{F}}} \cdot E = 1 + \ell(p) = 1 + a(p);$$

Case E not $\widetilde{\mathcal{F}}$-invariant

$$\mathrm{Tang}(\widetilde{\mathcal{F}}, E) = E^2 - T_{\widetilde{\mathcal{F}}} \cdot E = -1 + (\ell(p) - 1) = \ell(p) - 2 = a(p) - 1$$

Now, if X compact: $c_2(\tilde{X}) = c_2(X) + 1 \Rightarrow$

$$m(\tilde{\mathcal{F}}) = c_2(\tilde{X}) + T_{\tilde{\mathcal{F}}}^2 + T_{\tilde{\mathcal{F}}} . K_{\tilde{X}} = c_2(X) + 1 + T_{\mathcal{F}}^2 + (\ell(P)-1)^2 E^2 +$$
$$+ \left(\pi^* T_{\mathcal{F}} \otimes O_{\tilde{X}}(\ell(P)-1)E)\right) . \left(\pi^* K_X \otimes O_{\tilde{X}}(E)\right) =$$
$$= c_2(X) + 1 + T_{\mathcal{F}}^2 + (\ell(P)-1)^2 E^2 + T_{\mathcal{F}} . K_X + (\ell(P)-1) E^2 =$$
$$= \left[c_2(X) + T_{\mathcal{F}}^2 + T_{\mathcal{F}} . K_X\right] - \ell(P)^2 + \ell(P) + 1$$

Allora: $m(\tilde{\mathcal{F}}) = m(\mathcal{F}) - \ell(P)^2 + \ell(P) + 1$

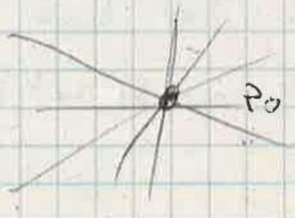
② $\mathbb{P}^2$:

Any foliatio $\mathcal{F}$ corresponds to a line bundle $O_{\mathbb{P}^2}(n) = T_{\mathcal{F}}$.

Definition)
The degree of $\mathcal{F}$ is $d(\mathcal{F}) := \mathrm{Tang}(\mathcal{F}, \ell)$, where ℓ is a line not $\mathcal{F}$-invariant.

It follows: $T_{\mathcal{F}} = O_{\mathbb{P}^2}(1 - d(\mathcal{F}))$, $N_{\mathcal{F}} = O_{\mathbb{P}^2}(2 + d(\mathcal{F}))$

Corollary) (more a side note)
If $d(\mathcal{F}) = 0 \Rightarrow$ Any non-invariant line ℓ is transverse to $\mathcal{F}$. $\Rightarrow P \notin \mathrm{Sing}(\mathcal{F})$, if $\ell_P \ni P$, and is tangent to P, then ℓ_P is $\mathcal{F}$-invariant.

$\mathcal{F}$ is therefore the radial foliation 

,

$\mathcal{F}$ given by, around $p_0 = (0,0)$, $z\frac{\partial}{\partial z} + w\frac{\partial}{\partial w}$.

In general)

$\forall\, d(\mathcal{F})$, $\mathcal{F}$ is locally given by

$$[P + zR]\frac{\partial}{\partial z} + [Q + wR]\frac{\partial}{\partial w}$$, with $P, Q \in \mathbb{C}[z,w]_{\leq d(\mathcal{F})}$ homogeneous

and $R \in \mathbb{C}[z,w]_{d(\mathcal{F})}$ homogeneous.

Also, $m(\mathcal{F}) = d(\mathcal{F})^2 + d(\mathcal{F}) + 1$.

③ $Bl_{\{P\}} \mathbb{P}^2$:

$\mathcal{F}_0$ foliation on $\mathbb{P}^2$, $\mathcal{F}$ corresponding foliation on $X = Bl_{\{P\}} \mathbb{P}^2$.

Let $d = d(\mathcal{F}_0)$, $l = l(P)$, $a = a(P)$, E = exceptional curve,

L = strict transform of line not containing P, so $L \in |\pi^* \mathcal{O}_{\mathbb{P}^2}(1)|$.

Then
$$T_{\mathcal{F}} = \mathcal{O}_X((1-d)L) \otimes \mathcal{O}_X((l-1)E)$$
$$N_{\mathcal{F}} = \mathcal{O}_X((2+d)L) \otimes \mathcal{O}_X(-lE)$$

$$\Rightarrow m(\mathcal{F}) = d^2+d+1 + l^2+l+1 = d^2+d+2-l^2+l.$$

Suppose $\mathrm{Sing}(\mathcal{F}) = \emptyset \Rightarrow \mathcal{F}_0$ has only P as singular point,

with $m(P) = d^2+d+1$.

There is only one option, that is: E is not $\mathcal{F}$-invariant,

and so $l = a+1$, $m(\mathcal{F}) = 0 \Rightarrow a^2+a = d^2+d+2$.

The only solution is $a=1, d=0 \Rightarrow \mathcal{F}_0$ is radial foliation.

($\mathcal{F}$ then is $\mathbb{F}_1 \longrightarrow \mathbb{P}^1$)

Theorem (Seidenberg thm)

Given any sing. point p of $\mathcal{F}$, $\exists$ finite sequence of blow-ups over p $\tilde{\pi}: \tilde{S} \to S$ s.t. $\tilde{\mathcal{F}}$ has only reduced sing. on $\tilde{\pi}^{-1}(p)$.

Proof (sketch)

Vedi carti su Blow-up, Chapter 2, Esempio (1)

let $\pi': S' \to S$ the blow-up along p, so we have $E' \mapsto p$

$$\sum_{i=1}^{k} m(p_i) = m(p) - \ell(p)^2 + \ell(p) + 1$$

where $p_1, \dots, p_k \in \mathrm{Sing}(\mathcal{F}') \cap E'$.

If $\ell(p) \geq 2$ (and in particular, if $a(p) \geq 2$), then $m(p_i) < m(p)$ $\forall i = 1, \dots, k$

[If $\ell(p) \geq 2 \Rightarrow -\ell(p)^2 + \ell(p) + 1 \leq -1$]

That is, the multiplicity decreses strictly at every step, but $m(\cdot) \in \mathbb{N}$, so after finitely many steps we are in the situation where $a(p) = 1$ $\forall$ p = singular point (which is not reduced)

So, we are in $\pi'': S'' \to S$.

Therefore, for all $p \in \mathrm{Sing}(\mathcal{F}'')$ (non reduced), $D(\mathcal{F}'')(p) \neq 0$.

$D :=$ $\lambda \in \mathbb{Q}^+$

The proof then divides into two cases
- D nonnilpotent → easier case.
- D nilpotent $\lambda_1 = \lambda_2 = 0$

$\boxtimes$

Remark

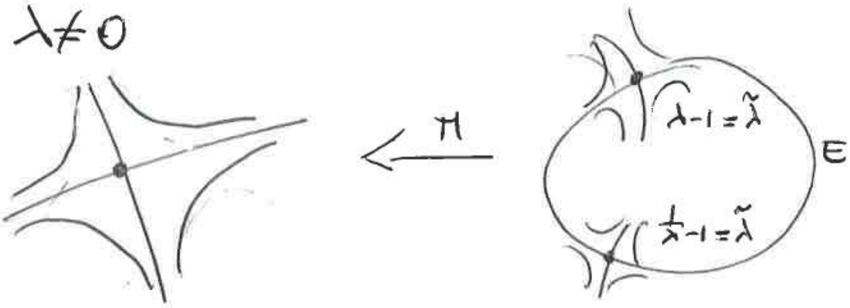
Blowing-up reduced sing. produces more reduced sing., so the result cannot be improved:

Let $p \in \mathrm{Sing}(\mathcal{F})$ reduced:

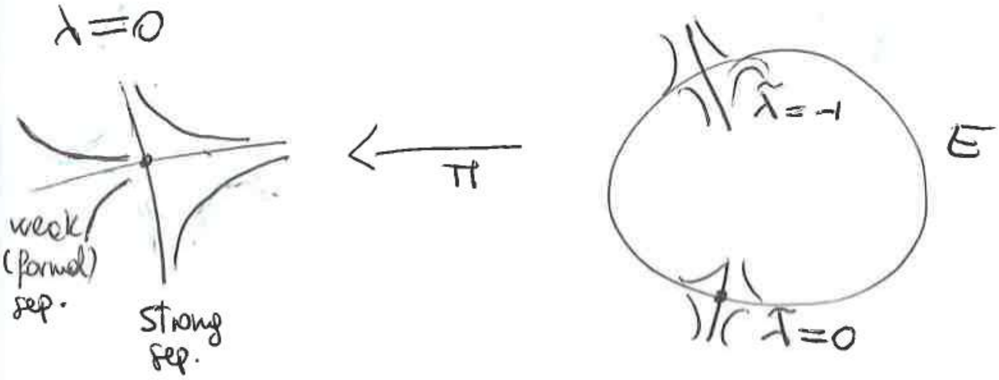
If p = nondegenerate, $\lambda \neq 0$, then E is $\tilde{\mathcal{F}}$-inv. and E contains 2 sing points ($E \cap \{$two separatrices$\}$), both reduced, nondeg. and with eigenvalues $\lambda - 1$ and $\frac{1}{\lambda} - 1$.

If p = saddle-node, $\lambda = 0$, then E is $\tilde{\mathcal{F}}$-inv. and contains 2 sing. points $\left(\begin{matrix} \text{saddle-node} \\ + \\ \text{nondeg. with } \lambda = -1 \end{matrix} \right)$

$\lambda \neq 0$



$\lambda = 0$



Proposition

$p \in \operatorname{Sing}(\mathcal{F})$ is dicritical iff $\exists\, \hat{\pi} : \tilde{S} \to S$ sequence of blowups over p and an irr. comp. E_0 of $\tilde{E} = \hat{\pi}^{-1}(p)$ that is not $\tilde{\mathcal{F}}$-invariant.

Proof

1) E_0 is not $\tilde{\mathcal{F}}$-invariant; Through the generic point of E_0, which is smooth, it passes a leaf $\tilde{C}$ and $\hat{\pi}(\tilde{C}) =: C$ is $\mathcal{F}$-invariant and contains p, i.e. C = separatrix.

True for the generic point of $E_0 \Rightarrow \exists$ inf. many separatrices through p.

2) $p \in \mathrm{Sing}(\mathcal{F})$ dicritical; let $p \in C$ be a separatrix. Its strict transform $\tilde{C}$ is again $\tilde{\mathcal{F}}$-invariant and contains the point $\tilde{p} = \tilde{C} \cap \tilde{E}$.

If $\tilde{E}$ entirely $\tilde{\mathcal{F}}$-invariant, then $\tilde{p} = \tilde{C} \cap \tilde{E}$ is singular (and $\tilde{C}$ = sep. of $\tilde{p}$).

Take $\hat{\pi}$ to be the sequence of blow-ups from Seidenberg thm.:

$\tilde{p}$ sing. $\Rightarrow \hat{p}$ = reduced sing. $\Rightarrow \hat{p}$ has finitely many (actually, $\overset{\leq}{\text{two}}$) separatrices.

But $\# \mathrm{Sing}(\hat{\mathcal{F}}) \cap \hat{\pi}^{-1}(p) < +\infty \Rightarrow p$ has finitely many separatrices $\times$

Therefore, $\tilde{E}$ must not be entirely $\tilde{\mathcal{F}}$-inv. $\boxtimes$

Blow-up of non-reduced sing. with $\lambda \in \mathbb{Q}^+$

$\boxed{\lambda \notin \mathbb{N}^+ \cup \frac{1}{\mathbb{N}^+}}$ (Nonresonant case)

$\mathcal{F}$ is linearizable around p and $V = n x \partial_x + m y \partial_y$, $\frac{n}{m} = \lambda$.

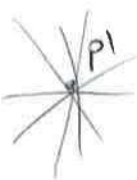
Suppose $n > m$.

- Blow up $p = (0,0) \rightsquigarrow p_1$ = reduced, Siegel domain, p_2 = nonreduced

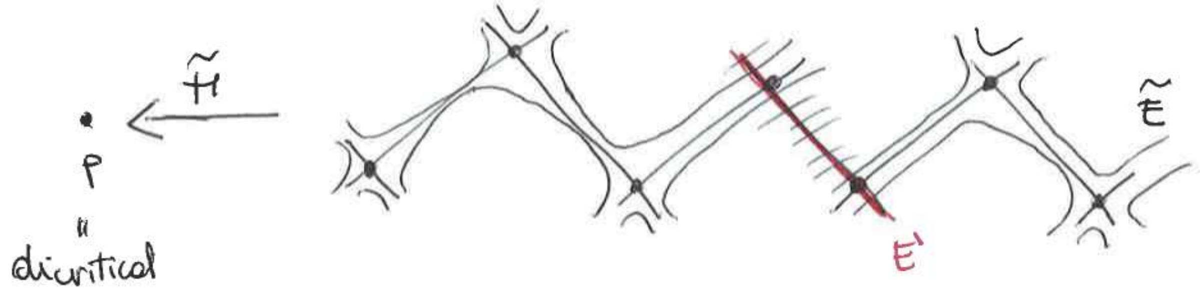
$\tilde{V} = n x \partial_x + (m-n) y \partial_y$ $\qquad \hat{V} = (n-m) x \partial_x + m y \partial_y$, $\frac{n-m}{m} \in \mathbb{Q}^+$

- Blow up $p_2 \rightsquigarrow$ after finite number of steps we are left with

p', $\hat{V} = x \partial_x + y \partial_y$



$\xrightarrow[\text{blowup } p']{}$ $\quad E'$ is not $\tilde{\mathcal{F}}$-inv.

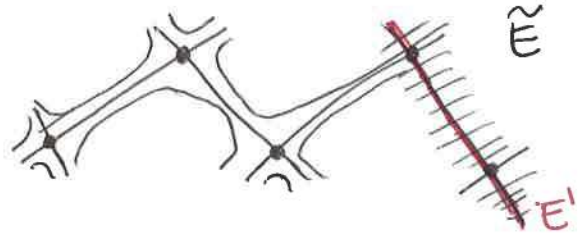


$\lambda \in \mathbb{N}^+ \cup \frac{1}{\mathbb{N}^+}$ | Resonant case |

$$V = x\partial_x + (nY + \varepsilon x^n)\frac{\partial}{\partial w} \quad , n = \lambda \text{ or } \lambda^{-1}, \ \varepsilon \in \{0,1\}$$

$\boxed{\varepsilon = 0}$

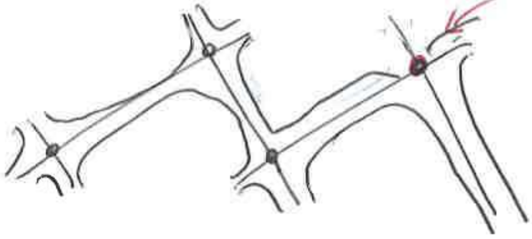
$\mathring{p} \xleftarrow{\hat{\pi}}$



$\overset{\shortparallel}{\text{dicritical}}$

saddle-node of multiplicity 2

$\boxed{\varepsilon = 1}$



$\mathring{p}$

$\overset{\shortparallel}{\text{nondicritical}}$