READING SEMINAR ON FOLIATIONS

30/10/2023

BASICS ON THE STRUCTURE OF FOLIATIONS ON SURFACES ①

§ I Definitions

Recall: X smooth / normal variety. A (singular holomorphic) *foliation* $\mathcal{F}$ on X is given by $0 \to T_{\mathcal{F}} \ (= \mathcal{F}) \to T_X$ s.t.

① $T_{\mathcal{F}}$ is a coherent sheaf which is *saturated*, i.e. $T_X / T_{\mathcal{F}}$ is torsion free ("maximality condition")

$\Longrightarrow$ T_X is reflexive, $\operatorname{codim} \operatorname{Sing}(\mathcal{F}) \geq 2$, $\operatorname{Sing} \mathcal{F} := \operatorname{Sing} X \cup \{x \in X$ ~~: $\mathcal{F} \subset T_X$~~ $\mathcal{F}$ not subbundle$\}$

② $T_{\mathcal{F}}$ is close under Lie bracket ("involutive distribution")

$\overset{\text{Frobenius}}{\Longrightarrow}$ $\forall x \in X - \operatorname{Sing}(\mathcal{F})$ $\exists$ hol. submersion $f : U \to W$, U nbg of x, $U \subseteq \mathbb{C}^n$, $W \subseteq \mathbb{C}^{n-r}$

and $\ker(df) = T_{\mathcal{F}}$

Rank of $\mathcal{F}$, $\operatorname{rk} \mathcal{F}$ is the rank of $T_{\mathcal{F}}$ as a sheaf.

From now on X ~~sm~~ smooth surface, $\operatorname{rk} \mathcal{F} = 1$.

$T_{\mathcal{F}}$ reflexive $\Rightarrow$ $T_{\mathcal{F}}$ locally free

So we have $0 \to T_{\mathcal{F}} \to T_X \to I_Z N_{\mathcal{F}} \to 0$

where $N_{\mathcal{F}}$ is a line bundle and $I_Z \subset \mathcal{O}_X$ ideal sheaf

~~Supported~~ s.t. $\operatorname{supp} I_Z = \operatorname{Sing}(\mathcal{F})$

Terminology:

$T_{\mathcal{F}}$: tangent bundle of $\mathcal{F}$, $K_{\mathcal{F}} = T_{\mathcal{F}}^*$ cotangent bundle / canonical bundle (Cartier divisor)

$N_{\mathcal{F}}$: normal bundle $N_{\mathcal{F}}^*$ co-normal bundle, $K_X = K_{\mathcal{F}} \otimes N_{\mathcal{F}}$

Rmk X normal surface, $T_{\mathcal{F}}$ reflexive $\Rightarrow \exists\, K_{\mathcal{F}}$ Weil divisor s.t. $T_{\mathcal{F}}^* \simeq \mathcal{O}_X(K_{\mathcal{F}})$.

Description
~~Interpretation~~ via vector fields

$0 \to T_{\mathcal{F}} \to T_X \to I_Z N_{\mathcal{F}} \to 0$ $\mathcal{F}$ is given by a section $s \in \Gamma(X, T_{\mathcal{F}}^* \otimes T_X)$, i.e. we have open covering $\{U_i\}$ s.t. (we can take a trivialisation $\{(U_i, v_i)\}$)

s on U_i is given by v_i vector field (with coefficient in $T_{\mathcal{F}}^*$)

and on $U_i \cap U_j$ $v_i = g_{ij} v_j$, $g_{ij} \in \mathcal{O}_X^*(U_i \cap U_j)$

are transition function of $T_{\mathcal{F}}^* = K_{\mathcal{F}}$.

$\mathrm{Sing}(\mathcal{F}) \cap U_i$ = zeroes of v_i (isolated)

Rmk: • we can always multiply by an invertible hol. funct.

(• if ~~v has~~ let $s \in \Gamma(X, T_{\mathcal{F}}^* \otimes T_X)$ ~~with~~ that s vanishes on a divisor $D = \{f=0\}$ then we can ~~multiply by~~ divide s by f and obtain a foliation (we are saturating s)

Same as considering s as a section of $T_{\mathcal{F}}^* \otimes T_X \otimes \mathcal{O}_X(-D)$)

Description via differential forms

Dually, we can consider $0 \to N^*_{\mathcal{F}} \to T^*X \to I_Z T^*_{\mathcal{F}} \to 0$ (with $T^*X = \Omega^1_X$)

and so a foliation is given by a global section of $N_{\mathcal{F}} \otimes T^*X$,

i.e. $\{(U_i, \omega_i)\}$ s.t. ω_i 1-forms and on

$U_i \cap U_j \quad \omega_i = h_{ij}\, \omega_j$, $h_{ij} \in \mathcal{O}^*(U_i \cap U_j)$ transition functions of $N_{\mathcal{F}}$.

(Again $\{$zeroes of $\omega_i = \mathrm{Sing}(\mathcal{F}) \cap U_i\}$ so isolated).

Examples (local) on $\mathbb{C}^2$

1) $\mathcal{F}$ given by $\frac{\partial}{\partial x}$ (dy)
smooth:

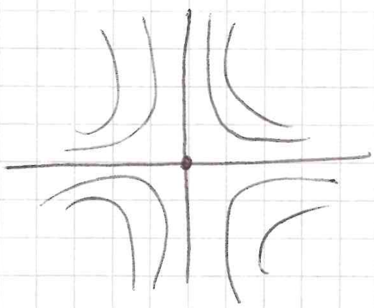
2) $\mathcal{F}$ given by $v = \lambda_1 x \frac{\partial}{\partial x} + \lambda_2 y \frac{\partial}{\partial y}$ $\quad (\omega = \lambda_1 x\, dy - \lambda_2 y\, dx)$

Integral curves $\gamma(t) = (c_1 e^{\lambda_1 t}, c_2 e^{\lambda_2 t})$

$\lambda_1 = 1$
$\lambda_2 = -1$

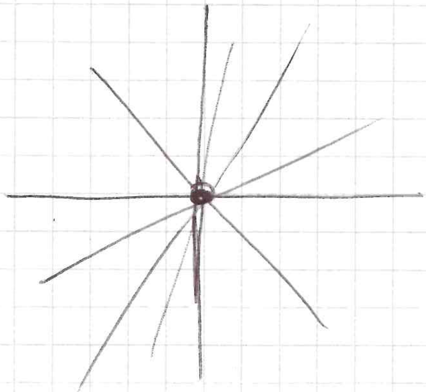
$v = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$

$\omega = x\,dy + y\,dx$



$\lambda_1 = 1$, $\lambda_2 = +1$ $\quad v = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$

radial foliation $\quad \omega = x\,dy - y\,dx$



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§ Singularities

$\mathcal{F}$ foliation on X, $p \in Sing(\mathcal{F})$. Take a local chart so that $\mathcal{F}$ is defined by a vector field $U \ni p = (0,0)$

$$v = A(x,y)\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y}$$

$A(0,0) = B(0,0) = 0$.

Def. p is a reduced singularity for $\mathcal{F}$ if $M = \begin{pmatrix} \frac{\partial A}{\partial x} & \frac{\partial A}{\partial y} \\ \frac{\partial B}{\partial x} & \frac{\partial B}{\partial y} \end{pmatrix}(0,0)$ has at least one $\neq 0$ eigenvalue λ_1 and $\lambda_2/\lambda_1 \notin \mathbb{Q}_+$.

Def. The multiplicity of $p \in Sing(\mathcal{F})$ is $\dim_{\mathbb{C}} \frac{\mathcal{O}_p}{\langle A,B \rangle}$

If X compact, set $m(\mathcal{F}) = \sum_{p \in Sing(\mathcal{F})} m(p)$

Prop. 2 $\mathcal{F}$ foliation on a compact surface X, then

$$m(\mathcal{F}) = T_{\mathcal{F}} \cdot T_{\mathcal{F}} + T_{\mathcal{F}} \cdot K_X + c_2(X)$$
$$= c_2(X) - T_{\mathcal{F}} \cdot N_{\mathcal{F}} \qquad (-N_{\mathcal{F}} = K_{\mathcal{F}} \otimes T_{\mathcal{F}})$$

Proof

$\mathcal{F}$ is given by a section $s \in \Gamma(X, K_{\mathcal{F}} \otimes T_X)$ and

$$m(\mathcal{F}) = \text{zeroes}(s) = c_2(K_{\mathcal{F}} \otimes T_X)$$
$$= c_2(T_X) + c_1(T_X)c_1(K_{\mathcal{F}}) + c_1(K_{\mathcal{F}})^2$$

↑ Bott-Tu, Thm 11.17

□

Rmk: $m(\mathcal{F}) = \underbrace{T_{\mathcal{F}} \cdot T_{\mathcal{F}} + T_{\mathcal{F}} \cdot K_X}_{\text{even}} + c_2(X)$

In particular, if $c_2(X)$ odd then $\mathcal{F}$ is singular.

§ Adjunction formulas

$$K_X = K_{\mathcal F} \otimes N^*_{\mathcal F}$$

Def An irreducible curve $C \subseteq X$ is $\mathcal F$-invariant if the inclusion $0 \to \mathcal F_{|C} \to T_{X|C}$ factors through

$$0 \to \mathcal F_{|C} \longrightarrow T_{X|C}, \qquad \mathcal F_{|C} \searrow T_C \nearrow T_{X|C}$$

(so generically C is a leaf)

Let C be an (compact) curve which is not $\mathcal F$-invariant (each comp. is), $p \in C$. The tangency order of $\mathcal F$ to C at p is

$$\operatorname{tang}(\mathcal F, C, p) := \dim_{\mathbb C} \frac{\mathcal O_p}{\langle f, v(f)\rangle}$$

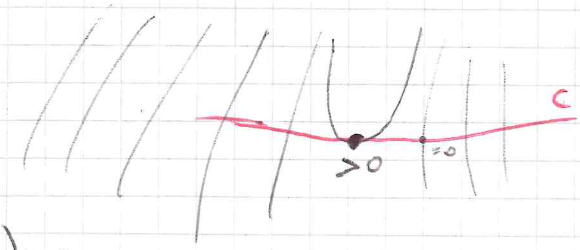
where $C = \{f = 0\}$ around p and v defines $\mathcal F$. $\operatorname{tang}(\mathcal F, C) = \sum_{p \in C} \operatorname{tang}(\mathcal F, C, p)$

Rmks:

1) Since $\mathcal F$ is not $\mathcal F$-invariant, $v(f)$ is not zero in a ngb of p (f and $v(f)$ not multiple of each other) and so

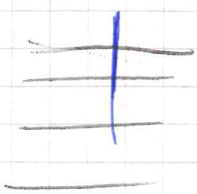
$$\dim_{\mathbb C} \mathcal O_p / \langle f, v(f)\rangle < +\infty$$

2) $\operatorname{tang}(\mathcal F, C, p) = 0 \iff v(f)$ is a unity in $\mathcal O_p \iff \mathcal F$ and C are transvers at p



Ex: $\mathcal F : \frac{\partial}{\partial x}$, $C = \{x = \text{const.}\}$

$\frac{\partial x}{\partial x} = 1 \qquad \operatorname{tang}(C, \mathcal F, p) = 0$



Proposition 2 Let $\mathcal{F}$ be a foliation on a surface X and $C \subseteq X$ a compact (irr.) curve not invariant by $\mathcal{F}$. Then

$$\sout{N_{\mathcal{F}} \cdot C =} \; T_{\mathcal{F}} \cdot C = C^2 - \operatorname{tang}(\mathcal{F}, C)$$

$$\left(N_{\mathcal{F}} \cdot C = \chi(C) + \operatorname{tang}(\mathcal{F}, C) \quad \text{, where } \chi(C) = -K_X \cdot C - C^2 \right)$$

Proof Take ~~$\{U_i\}$ covering of X, v_i vector field~~

$\{(U_i, v_i)\}$ describing $\mathcal{F}$ and $\{(U_i, f_i)\}$ describing C

so that on $U_i \cap U_j$ $\quad v_i = g_{ij} v_j$, $\quad f_i = h_{ij} f_j$

where g_{ij} are transition functions of $T^*_{\mathcal{F}}$

h_{ij} " " " of $\mathcal{O}_X(C)$

By Leibnitz rule $\quad v_i(f_i) = g_{ij} v_j (h_{ij} f_j) = g_{ij} h_{ij} v_j(f_j) + g_{ij} f_j v_j(h_{ij})$

$$= g_{ij} h_{ij} v_j(f_j) \quad \text{on } C$$

So $\{(U_i|_C, v_i(f_i)|_C)\}$ gives a section of $T^*_{\mathcal{F}} \otimes \mathcal{O}_X(C)|_C$

This vanishes with the ~~right~~ order $\operatorname{tang}(\mathcal{F}, C, p)$ at ~~the~~ any tangency point p.

So $\operatorname{tang}(\mathcal{F}, C) = \deg(T^*_{\mathcal{F}} \otimes \mathcal{O}(C))|_C = T^*_{\mathcal{F}} \cdot C + C^2$ $\quad \square$

Rmk If C is everywhere transverse to $\mathcal{F}$, we obtain

$T_{\mathcal{F}} \cdot C = C^2$, which is obvious since $T_{\mathcal{F}}|_C \simeq N_C$

Consider now an irr. curve C invariant by $\mathcal{F}$. Let $p \in C$, f a local equation for C around p, $C = \{f = 0\}$ and ω a 1-form generating $\mathcal{F}$ at p. $\omega = B\,dx - A\,dy$, $v = A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y}$

Lemma Locally $\exists$ 1-form η and hol function g, h s.t.

$$(*) \quad g\omega = h\,df + f\eta \quad . \qquad (h, f \text{ coprime, i.e. } h \text{ non-vanishing} \equiv_C 0)$$

Proof C $\mathcal{F}$-invariant $\Rightarrow$ $v(f) = kf$ ~~~~ k hol. function

$\Rightarrow A\frac{\partial f}{\partial x} + B\frac{\partial f}{\partial y} = kf$

Set $g = \frac{\partial f}{\partial y}$, $h = -A$ and $\eta = k\,dx$.

$\square$

(~~Nb~~ **Rmk** It's $\Leftrightarrow$, ~~because~~ using that $df \wedge \omega = -v(f)\,dx \wedge dy$
and $(*) \Rightarrow h\,df \wedge \omega = -f\eta \wedge \omega$ and so $v(f) \in (f)$)

Def. Define $Z(\mathcal{F}, C, p)$ as the vanishing order of $\frac{h}{g}\big|_C$ at p. $Z(\mathcal{F}, C) := \sum Z(\mathcal{F}, C, p)$

Rmk $\frac{h}{g}\big|_C$ does not depend on the choice of g, h and η.

It changes by a nowhere vanishing hol. function changing ω and f.

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~~Rmk.~~

Rmk. Assume p is regular in C. So $C = \{y=0\}$, $f = y$.

Since $A \frac{\partial}{\partial x} y + B \frac{\partial}{\partial y} y = B$ must be in the ideal of (y), we have that $y \mid B$ (so we get $v = A \frac{\partial}{\partial x} + C y \frac{\partial}{\partial y}$)

$\omega = -A\,df + B\,dx$

so we are looking at the order of vanishing of $A_{|C}$ in p, in particular $Z(\mathcal{F}, C, p) \geq 0$ and if $p \notin \mathrm{Sing}(\mathcal{F})$ $Z(\mathcal{F}, C, p) = 0$

Examples

1) $\omega = x\,dy + y\,dx$, $C = \{y = 0\}$ $(\omega = x\,df + y\,dx)$

$h = x$, $g \equiv 1$

So $Z(\mathcal{F}, C, p(0,0)) = 1$

2) $\omega = 2y\,dx - 3x\,dy$, $C = \{f = x^2 - y^3 = 0\}$

Then $Z(\mathcal{F}, C, p = (0,0)) = -1$

Prop. 3 $\mathcal{F}$ foliation on a surface X, C be an (irr) curve $\mathcal{F}$-invariant. Then

$$N_{\mathcal{F}} \cdot C = C^2 - Z(\mathcal{F}, C)$$
$$K_{\mathcal{F}} \cdot C = \chi(C) - Z(\mathcal{F}, C).$$

Proof Same as for $\mathrm{tang}(\mathcal{F}, C)$, but using forms and the decomposition $(*)$. $\square$

§ Global examples

Recall: $K_{\mathcal{F}} = K_X \otimes N_{\mathcal{F}}$

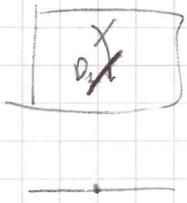
1) Fibrations

X compact surface, $\pi : X \to C$ fibration, C curve

$K_{X/C} = K_X - \pi^* K_C$ relative canonical bundle

This induces a foliation, $\mathcal{F} : \ker(d\pi) \subseteq T_X$

Take $p \in C$, z local coordinate at p. $\pi^* p = \sum_{i=1}^{k} m_i D_i$



$m_1 > 1$, locally around D_1, $\pi : (x, y) \mapsto x^m$

$\pi^* dz = m\, x^{m-1} dx$. Consider $\dfrac{\pi^* dz}{\prod g_i^{m_i - 1}}$ $\quad \{g_i = 0\} = D_i$

$\Rightarrow N^*_{\mathcal{F}} = \pi^* K_C \otimes \mathcal{O}_X\left(\sum (m_i - 1) D_i\right)$

In words $\pi^* dz$ is a local section of $\pi^*(K_C)$ and of $N^*_{\mathcal{F}}$ vanishing with order $(m_i - 1)$ along D_i

$\Rightarrow T^*_{\mathcal{F}} = K_{X/C} \otimes \mathcal{O}_X\left(\sum (1 - m_i) D_i\right)$.

($T^*_{\mathcal{F}} = K_{\mathcal{F}}$)

2) Foliations on $\mathbb{P}^2$

$X = \mathbb{P}^2$. We define $d = d(\mathcal{F}) := \operatorname{tang}(\mathcal{F}, L)$ the degree of $\mathcal{F}$ where L is a general line in $\mathbb{P}^2$.

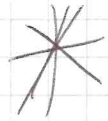
$K_{\mathcal{F}} \cdot L = -L^2 + \operatorname{tang}(\mathcal{F}, L) = -1 + d$ $\qquad c_2(\mathbb{P}^2) = 3$

$\Rightarrow K_{\mathcal{F}} \sim \mathcal{O}(d-1)$, $\quad m(\mathcal{F}) = d^2 + d + 1$

$d = 0$

$K_{\mathcal{F}} \sim \mathcal{O}(-1)$

any non-$\mathcal{F}$-invariant line is transverse to $\mathcal{F}$.
Since there are tangent lines to any leaf of the foliation
$\Rightarrow \mathcal{F}$ is a pencil of line
$\Rightarrow$ locally around $(0,0)$



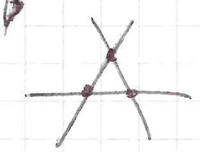
$x \dfrac{\partial}{\partial x} + y \dfrac{\partial}{\partial y}$

(projection from a point)

$d = 1$

$K_{\mathcal{F}} \sim \mathcal{O}_{\mathbb{P}^2}$

Since $m(\mathcal{F}) = 3$, $\exists$ a singular point p.
Any (other) line not $\mathcal{F}$-invariant is such that $\operatorname{tang} = 1$, but $\exists$ $L \ni p$ s.t. $\operatorname{tang}(\mathcal{F}, L, p) \geq 2$ (why?). Put this line at infinity



$\Rightarrow$ locally $A(x,y) \dfrac{\partial}{\partial x} + B(x,y) \dfrac{\partial}{\partial y}$ $\quad A(x,y), B(x,y)$ linear