EXAMPLES

- $f : X \longrightarrow Y$ $\qquad T_{X/Y} \subseteq T_X$ is A FOLIATION [alg. integrable]

  $f : X \dashrightarrow Y$

- vector fields: in $\mathbb{C}^2$, take

  $v = f(x,y)\partial_x + g(x,y)\partial_y \rightsquigarrow \mathcal{F} = \mathcal{O}_X \cdot v$

  SATURADNESS $\iff$ $f(x,y) = g(x,y) = 0$ has codim 2 in $\mathbb{C}^2$

  LIE BRACKET $\iff$ $[v,v] \subseteq \mathcal{O}_{\mathbb{C}^2} \cdot v$

  IMMEDIATE: $[v,v] = 0$

- DESTABILIZING BUNDLES: X smooth / normal

  $\mathcal{F} \subseteq T_X$ not stable

  max'l destabilizing

  - SATURADNESS ✓ By max'lity
  - $[\mathcal{F},\mathcal{F}] \longrightarrow T_{X/\mathcal{F}}$ stability

    $\Lambda^2 \mathcal{F} \longrightarrow T_X/\mathcal{F}$ is 0-map.

THM (Miyaoka) X is not uniruled, smooth, projective.

Then Ω^1_X is generically semi-positive

[i.e., fix $H_1, \dots, H_{\dim X - 1}$ ample divisors
$m_1, \dots, m_{\dim X - 1}$ $m_i \gg 0$ integers

$C = \widehat{H}_1 \cap \dots \cap \widehat{H}_{\dim X - 1}$ $\quad \widehat{H}_i \in |m_i H_i|$ general

generically semipositive $\overset{\text{DEF}}{\Longleftrightarrow}$ $\Omega^1_X|_C \twoheadrightarrow \mathcal{G}$ torsion free
$\deg_C \mathcal{G} \geq 0$]

Equivalently: $T_X|_C$, $\underbrace{\mathcal{G}' \subseteq T_X|_C \quad \deg \mathcal{G}' \leq 0}$.

THM [Miyaoka, Shepherd-Barron]

X smooth projective. Assume that

$\boxed{\exists\ \mathcal{E} \subseteq T_X}$ ~~saturated~~ s.t.

$c_1(\mathcal{E}|_C) > 0$. Then $\exists\ \mathcal{F} \subseteq T_X$ saturated

[satisfies the L.B. property] s.t. $c_1(\mathcal{F}|_C) > 0$

& R rat'l curve through a generic $x \in X$ s.t.:

- R is smooth @ $x \in X$
- $T_{R,x} \subseteq \mathcal{F}_x$
- B&B inequality: $\underline{H} \cdot R \leq 2\dim X \left(\underline{H} \cdot C\right) / c_1(\mathcal{F}) \cdot C$

More general Miyaoka's THM

$\mathcal{F} \subseteq T_X$ is a foliation which is alg. integrable w/ RCC leaves

$$\mathcal{F} \subseteq T_X$$

$$\underbrace{\mathcal{F}'}_{\text{algebraically integrable}} \quad \underbrace{\subseteq}_{\text{subfoliation}} \quad \mathcal{F}$$

$f : X \dashrightarrow Y$, $\mathcal{H} = \mathcal{F}/\mathcal{F}'$

$\mathrm{Ker}(df) = \mathcal{F}'$

if $\mathcal{F}'$ is max'lly alg. int. $\Rightarrow$ $\mathcal{H}$ is completely transc.

[AD, Fano foliations]

Green - Griffiths : Let X be a smooth, projective variety of general type

$$\left[\operatorname{Kod}(K_X) = \dim X \right]$$

$\Updownarrow$

K_X is big

Then $\exists\ Z \subsetneq X$ alg. subvariety s.t.

$$\forall\ f : \mathbb{C} \xrightarrow[\text{non-const.}]{\text{hol.}} X \qquad f(\mathbb{C}) \subseteq Z$$

[In particular, $\nexists$ entire curves w/ Zar. dense image]

KNOWN RESULTS: in dim 2 CONJ HOLDS for a large class of gen'l type surfaces ($13\cancel{9}\,c_1^2(X) > 9\,c_2(X)$)

[Lu-Yau, Dem, GG]: $\exists$ only finitely many rat'l & elliptic curves on X gen'l type surface

COUNTEREX. TO GG CONJ. will come from Zariski dense entire curves.

THM 1 [McQ] X smooth gen'l type surface, $c_1^2(X) > c_2(X)$

$f : \mathbb{C} \to X$ entire curve [dense image]

$\implies \exists\ Z \xrightarrow{\pi} X$, Z smooth gen'l type projective,

$\wedge$ $\boxed{\exists\ \mathcal{F} \subseteq T_Z}$ s.t.

(1). $\exists$
$$\begin{tikzcd} & Z \arrow[d, "\pi"] \\ \mathbb{C} \arrow[ur, "g"] \arrow[r, "f"'] & X \end{tikzcd}$$
$\pi \circ g = f$

(2) g is tangent to $\mathcal{F}$ ($g(\mathbb{C})$ is a generically leaf of $\mathcal{F}$)

To prove $GGV^{\text{in dim}=2}$ one needs to prove

$\nexists$ Zariski dense entire curves

$\Uparrow$ [M^{c}Q]

$\nexists$ Zariski dense entire leaves on a gen'l type surface

<u>THM 2</u> [M^{c}Q] Let Z be a smooth gen'l type. Assume $\exists\, \mathcal{F} \subseteq T_Z$. Then $\nexists$ $g : \mathbb{C} \to Z$ dense entire leaves for Z.

X smooth proj surface, $\mathcal{F} \subseteq T_X$ foliation.

How to classify $(X, \mathcal{F})$? [in analogy w/ Enriques, Sev., Kodaira class.]

1. Understand how to transform foliations under birat'l maps [EASY]

2. Define nice classes of foliated sing's
[Resolution does not hold for foliations]
[We'll look also at MMP-type sing's for foliation]

3. Run the analogous of the MMP for $(X, \mathcal{F})$ w/ nice singularities & show that this is well-behaved, and terminates w/ good outc.