READING GROUP ON FOLIATIONS

1. INTRODUCTION & MOTIVATIONS

UNIV. OF MILAN

AY 2023–24

23.10.2023

FOLIATION: $\mathcal{F} \underset{\text{subsheaf}}{\subseteq} T_X$ [X normal variety] smooth for the time being

such that

① $T_X / \mathcal{F}$ is torsion free

② $[\mathcal{F}, \mathcal{F}] \subseteq \mathcal{F} \iff \underset{\text{skew symmetric}}{[-,-]} : \mathcal{F} \otimes \mathcal{F} \xrightarrow{\mathcal{O}_X\text{ - linear morphism}} T_X / \mathcal{F}$ is the 0 map.

SINGULARITIES: $x \in X^{sm}$ $\quad \mathcal{F}|_x \subseteq T_X|_x$

$$\mathrm{Sing}(\mathcal{F}) = X^{sing} \cup \{ x \in X^{sm} \mid \mathcal{F} \text{ around } x \in X \text{ is } \underline{\underline{\text{not}}} \text{ a subbundle} \}$$

THM [Frobenius] X smooth variety, $\mathcal{F} \subseteq T_X$ smooth foliation

$\Updownarrow$

$\mathrm{sing}(\mathcal{F}) = \phi$

Then, $\forall x \in X$, $\exists\, U \underset{x}{\overset{\psi}{}} \subseteq X$ analytic neighborhood

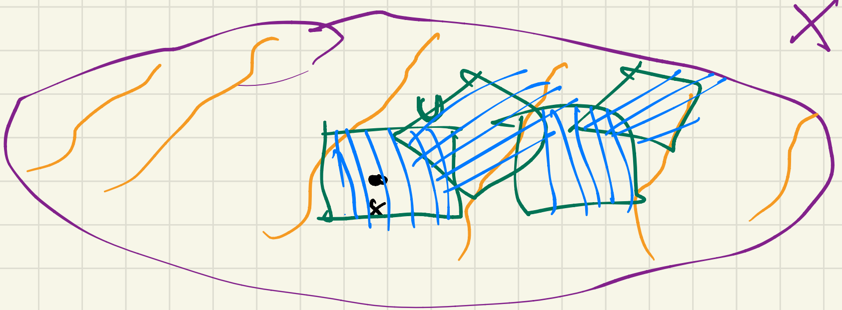
& $\exists\, f : U \subseteq \mathbb{C}^{\dim X} \longrightarrow W \subseteq \mathbb{C}^{r}$ surjective & smooth [holomorphic submersion]

$\mathrm{rk}(\mathcal{F}) = \dim X - r$

and $\mathrm{Ker}(df) = \mathcal{F}|_U$

$\Updownarrow$

$T_{U/W}$ $\mathrm{Ker}(df)$ is a distribution foliation $\subseteq T_X|_U$ of rank $\dim X - r$



Local leaves on U of $\mathcal{F}$: $\overset{\shortparallel}{\text{fibers}}$ of f

Leaves of $\mathcal{F}$: max'l analytic contin. of the local leaves.

REMARKS • To get the local integrals of $\mathcal{F}$ we need to work in the analytic cat. & so the leaves of a foliation will be analytic (most of the time they'll be transcendental, i.e., far from being alg.)

$\mathcal{F} \subseteq T_X$ a smooth foliation

$F \subseteq X$ a leaf of $\mathcal{F}$

Q: $\overline{F}^{\,zar}$?

Examples: A abelian surface, $\mathcal{F}$ a linear foliation

$\mathcal{F} \subseteq T_A = \mathcal{O}_A^{\oplus 2}$ $\quad [\,\mathcal{O}_X \longrightarrow \mathcal{O}_A^{\oplus 2}\,]$

$$\overline{F}^{\,zar} \begin{cases} A & \text{the leaf dense} \\ 1 \text{ dim'l} & \longleftrightarrow \mathcal{F} \text{ is alg. integrable} \overset{DEF}{\Leftrightarrow} \text{the leaves are algebraic} \end{cases}$$

ALGEBRAICALLY INTEGRABLE: $\mathcal{F} \subseteq T_X$ is alg. int. if the leaves of $\mathcal{F}$ are algebraic

REM. $\mathcal{F}$ is alg. int. $\iff$ $\exists$ $X \dashrightarrow Z$ whose fibers are the closure of the leaves of $\mathcal{F}$.

REM. $\left\{ \underset{\mathcal{F}}{\text{alg. int}} \right\} \subsetneq \left\{ \underset{\mathcal{F}}{\text{foliations}} \right\}$

was already for smooth foliations