Hyperbolicity, MMP and subadjunction

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Singularities

- A log pair (X, Δ) is an ordered pair given by a normal projective variety X and a Weil divisor Δ, whose coefficients vary in between 0 and 1, such that $K_X + \Delta$ is $\mathbb{R}$-Cartier.
- Let (X, Δ) be a log pair. Then we say that (X, Δ) has **log canonical** (in short lc) singularities if for every log resolution $\pi : Y \to X$, when we write the formula
$$K_Y = \pi^*(K_X + \Delta) - \pi_*^{-1}\Delta + \sum_i a_i E_i, \quad (1)$$
then $a_i \geq -1, \forall i$. The E_i's are the irreducible components of the exceptional divisor of π.
- For an lc pair (X, Δ), the **non klt locus of** Δ, $\mathrm{Nklt}(\Delta)$, is defined as the image under π of the irreducible components of $-\pi_*^{-1}\Delta + \sum_i a_i E_i$ of coefficient -1.
- A **lc center** is the image under π of an irreducible component of any intersection of irreducible divisors of coefficient 1 in $-\pi_*^{-1}\Delta + \sum_i a_i E_i$.
- $\mathrm{Nklt}(\Delta)$ carries a natural **stratification** given by the lc centers of Δ.
General idea: properties of a lc divisor Δ should be read off $\mathrm{Nklt}(\Delta)$ and its stratification.
- Given an lc pair (X, Δ), the MMP aims at obtaining (algorithmically) a new lc pair (X', Δ'), where X and X' are birational and the geometry of Δ' is better behaved than that of Δ. E.g., starting with a variety X, one would like to find a birational model X' s.t. either $K_{X'}$ is nef or X' has a fibre space structure.
- One crucial task is to determine what kind of **positivity** properties log divisors feature.
Cone Theorem[[1]] *Let (X, Δ) be a lc pair. Then, there are countably many rational curves $C_j \subset X$, with $0 \leq -(K_X + \Delta) \cdot C_j \leq 2 \dim X$ and*
$$\overline{NE}(X) = \overline{NE}(X)_{(K_X+\Delta)\geq 0} + \sum_i \mathbb{R}^+[C_j]$$
Thus, if (X, Δ) is a pair with lc singularities and X does not contain rational curves, then $(K_X + \Delta)$ is nef. However, such condition is a rather strong one.

Mori hyperbolicity

- **Question:** *What kind of geometric properties should Δ have in order for $K_X + \Delta$ to be **nef**?*
- In [3], Lu and Zhang introduced the notion of Mori hyperbolicity, which generalizes as follows to the case of a lc pair (X, Δ).
Definition 1 [cf. [3]] *A lc pair (X, Δ) is **Mori hyperbolic** if both $X \setminus \mathrm{Nklt}(\Delta)$ and $W \setminus \{W' \mid W'$ lc center, $W' \subset W\}$, for any lc center W, do not contain algebraic curves whose normalization is $\mathbb{C}$ or $\mathbb{P}^1$.*
- It is natural to expect, that under such hypotheses, using the Cone Theorem, one should be able to show:

Main Theorem
Let (X, Δ) be a Mori hyperbolic log canonical pair. Then $K_X + \Delta$ is nef

This is proved in [4].
- It is not too hard to prove this statement in the case of either a log smoooth or a dlt pair. The main ingredients are:
1. **Shokurov's Connectedness Theorem** *Let $\pi : X \to Y$ be a contraction of normal varieties and Δ a lc divisor on X such that $-(K_X + \Delta)$ is π-big and nef. Then* $\mathrm{Nklt}(\Delta)$ *is connected in the neighborhood of every fibre of π.*
2. **KV Vanishing** *Let $\pi : X \to Y$ be a contraction of normal varieties, Δ a klt divisor on X, M a Cartier divisor on X, with M $K_X + \Delta + N$ and N is π-big and nef. Then, $R^i\pi_*\mathcal{O}_X(M) = 0, \forall i > 0$.*
- Existence of **dlt modifications** and the two theorems above imply the following rather general statement.
Proposition 1 Let (X, Δ) be a Mori hyperbolic log pair. Then $K_X + \Delta$ is nef if and only if it is nef when restricted to its non klt locus.
- Proposition 1 suggests that one should work by induction on the strata of $\mathrm{Nklt}(\Delta)$. In the lc case, nonetheless, the structure of the stratification of $\mathrm{Nklt}(\Delta)$ is more complicated and one has to carry out a more refined analysis. The fundamental tool is the **canonical bundle formula**.

Subadjunction for lc pairs

- When (X, Δ) is Mori Hyperbolic and W is a minimal lc center, then $K_X + \Delta$ is nef along W. This is a consequence of **Kawamata subadjunction**.
- When W is no more minimal, take a dlt modification, i.e. a map $\pi : (Y, \Delta_Y) \to (X, \Delta)$, $K_Y + \Delta_Y = \pi^*(K_X + \Delta)$, (Y, Δ) is dlt and only divisors of **log-discrepancy 0** are extracted. Consider a minimal lc center S dominating W.
- **Inductive step:** $K_X + \Delta$ is nef along W, given that it is already nef along all the lc centers contained in W.
- Take the Stein factorization of the map $S \to W$
$$\pi_{|S} : S \xrightarrow{\pi_S} W_S \xrightarrow{\mathrm{spr}_W} W. \quad (2)$$
Results of Kollár allow to substitute W with W_S and $\mathrm{Nklt}(\Delta)$ with its preimage. The advantage is that now we are dealing with a **fibration**, π_S, and a **trivial divisor** over W_S, $(K_Y + \Delta_Y)_{|S} = K_S + \mathrm{Diff}_S^*\Delta$.
- As nefness is our goal, $(K_X + \Delta)_{|W}$ can be substituted with $(K_X + \Delta)_{|W} + \epsilon A$, $\epsilon \ll 1$, for A an ample divisor on W.
- Going to higher birational models of S and W
$$\begin{array}{ccc} (S, \Delta_S) & \xleftarrow{r_{S'}} & (S', \Delta_{S'}) \\ \downarrow{\scriptstyle \pi_S} & & \downarrow{\scriptstyle \pi_{S'}} \\ W & \xleftarrow{\quad r \quad} & W'. \end{array}$$
we can arrange that:
1. $r^*((K_X + \Delta)_{|W} + \epsilon A) = K_{W'} + \mathbb{B}_{W'} + \mathbb{M}_{W'}$;
2. $(W', \mathbb{B}_{W'})$ is sublc and snc, $\mathbb{M}_{W'}$ is nef;
3. $r(\mathrm{Nklt}(\mathbb{B}_{W'})) = \cup\{Z \mid W \supset Z \text{ lc center}\}$
- Perturbing $\mathbb{B}_{W'}$ with a relative anti-ample effective divisor E supported on the exceptional locus and pushing forward to W, we construct a boundary Δ' on W proving the following generalized weak form of subadjunction.
Theorem 1 Let (X, Δ) be an lc divisor and W an lc center. Then there exists a boundary Δ' on W^ν s.t. $K_{W^\nu} + \Delta' = (K_X + \Delta)_{|W^\nu}$ and there is a log resolution of (W^ν, Δ') on which we can arrange for the pull back of Δ' to have coefficients arbitrarily close to 1.
- Using Δ' and Proposition 1, it is then easy to complete the proof of the inductive step just by passing to a dlt modification of (W, Δ').

Ampleness

Theorem Let (X, Δ) be a -factorial dlt Mori hyperbolic pair. Suppose $K_X + \Delta$ is big. The following are equivalent:
1. $K_X + \Delta$ is ample;
2. $\lfloor \Delta \rfloor \cap \mathbb{B}_+(K_X + \Delta) = \emptyset$;
3. $Z \nsubseteqq \mathbb{B}_+(K_X + \Delta)$, for any stratum of $\mathrm{Nklt}(\Delta)$;
4. $K_X + \Delta$ is log big, i.e. it is big when restricted to any of stratum of $\mathrm{Nklt}(\Delta)$;
5. $(K_X + \Delta|_Z)^{\dim Z} > 0$, for any stratum of $\mathrm{Nklt}(\Delta)$;
6. $K_X + \Delta$ is ample when restricted to $\mathrm{Nklt}(\Delta)$.

The non lc case

- In Proposition 1, there are no hypotheses on the singularities of (X, Δ). That is possible since we can extend the definition of Mori hyperbolicity to arbitrary singularities, just by removing the locus where the singularities are non lc. That, existence of dlt modifications (of the support) of Δ and the theory of **quasi-log varieties** suggest that an inductive approach, analogous to the one carried out in the lc case, should be possible for arbitrary singularities.
- In order to generalize the above results to the non lc case, the only missing piece seems to be a stronger version of the classical Bend and Break Lemma. The needed result should allow to deform curves into sums of rational cycles, keeping the intersection numbers with some divisors under control.

References

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