The geometry of rational curves on some classes of varieties

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Research interests

My research interests are mainly focused on the study of **birational geometry** of algebraic varieties over $\mathbb{C}$ and especially the theory of the **Minimal Model Program**. More particularly, I am interested in question regarding the existence of rational curves and their interaction with the global geometry of varieties (e.g., the existence of rational curves on CY 3folds, positivity questions for log pairs), the birational geometry of Calabi Yau varieties (in particular questions regarding and consequences of the so-called **Cone Conjecture**) and criteria to determine whether a variety has special geometry (such as rationality, unirationality, toricness).

Hyperbolicity and MMP

One of the main intuitions of modern birational geometers is that instead of working with a fixed (normal) variety, X, one should focus on pairs given by X and a log divisor, Δ, i.e. a Weil divisor whose coefficients vary in between 0 and 1 and $K_X + \Delta$ is $\mathbb{R}$-Cartier. This approach allows one to discuss the singularity of X (and Δ) from a divisorial point of view.

Definition 1. *Let (X, Δ) a pair as defined above. Then we say that (X, Δ) has* **log canonical***, in short lc (resp.* **Kawamata log terminal***, or klt) singularities if for every log resolution $\pi : Y \to X$, when we write the formula*

$$K_Y = \pi^*(K_X + \Delta) - \pi_*^{-1}\Delta + \sum_i a_i E_i, \qquad (0.1)$$

then $a_i \geq -1, \forall i$ (resp. $a_i > -1$ and $\lfloor \Delta \rfloor = 0$), where the E_i's are the irreducible components of the exceptional divisor for π and $E = \sum_i E_i$.

In the case of an lc pair (X, Δ) one defines a sub variety of X, the non klt locus of Δ, $\mathrm{Nklt}(\Delta)$, as the image under π of those (irreducible) components of $-\pi_*^{-1}\Delta + \sum_i a_i E_i$ of coefficient -1. $\mathrm{Nklt}(\Delta)$ carries a stratification given by the images of all the possible intersections of these components.

Now, the basic idea of the MMP is that, starting from a given pair (X, Δ), there should be an algorithmic way to produce pairs (X', Δ'), where X and X' are birational and the geometry of Δ' is better behaved than that of Δ. For example, in the classical setting, one starts with a variety X and would like to find a birational model X' s.t. either $K_{X'}$ is nef or X' has some kind of fibre space structure. In view of this, it is crucial to determine what kind of positivity properties log divisors feature.

Theorem 2 (**Cone Theorem**, [1])**.** *Let (X, Δ) be a lc pair. Then, there are countably many rational curves $C_j \subset X$, with $0 \leq -(K_X + \Delta) \cdot C_j \leq 2 \dim X$ and*

$$\overline{NE}(X) = \overline{NE}(X)_{(K_X+\Delta)\geq 0} + \sum_i \mathbb{R}^+[C_j]$$

Thus, if (X, Δ) is a pair with mild singularities and X does not contain rational curves, then $(K_X + \Delta)$ is nef.

However, the condition that X does not contain rational curves is a rather strong one.

Question. *What kind of geometric properties should Δ have in order for $K_X + \Delta$ to be* **nef***?*

In [3], Lu and Zhang introduced the notion of Mori hyperbolicity, which generalizes as follows to the case of a lc pair (X, Δ).

Definition 3 ([3], [4])**.** *A lc pair (X, Δ) is* **Mori Hyperbolic** *if every open stratum of the (open) stratification on $\mathrm{Nklt}(\Delta)$ does not contain algebraic curves whose normalization is $\mathbb{C}$ or $\mathbb{P}^1$.*

It is natural to expect, that under such hypotheses, using the Cone Theorem, one should be able to show that $K_X + \Delta$ is nef. In fact, it is rather easy to prove this statement in the case of either a log smoooth or a dlt pair. Using that and dlt modifications one can immediately prove the following rather general statement.

Proposition 4 ([4])**.** *Let (X, Δ) be a Mori hyperbolic log pair. Then $K_X + \Delta$ is nef iff it is nef when restricted to its non klt locus.*

In the lc case, nonetheless, the structure of the stratification of $\mathrm{Nklt}(\Delta)$ is more complicated and one has to use a more refined analysis. By induction on the strata of $\mathrm{Nklt}(\Delta)$ and using the machinery of the canonical bundle formula, one can prove that, in the Mori Hyperbolic case, $K_X + \Delta$ is nef on the non-klt locus and hence on X.

Theorem 5 ([4])**.** *Let (X, Δ) be a Mori hyperbolic log canonical pair. Then $K_X + \Delta$ is nef.*

If one is interested in the ampleness of $K_X + \Delta$, then, in analogy, with Proposition 4, one can prove the following Nakai-type result.

Theorem 6 ([4])**.** *Let (X, Δ) be a Mori hyperbolic dlt pair. Suppose $K_X + \Delta$ is big. Then $K_X + \Delta$ is ample iff it is ample when restricted to $\mathrm{Nklt}(\Delta)$ iff $K_X + \Delta$ is log big.*

In order to generalize the above results to the non lc case, the only missing piece seems to be a stronger version of the classical Bend and Break Lemma. The needed result should allow to deform curves into sums of rational cycles, keeping the intersection numbers with some divisors under control.

References

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[4] R. Svaldi, *On Mori Hyperbolicity for log pairs*, in preparation.