Recent progress on the birational geometry of foliations on threefolds

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We will always work over $\mathbb{C}$.

A foliation on a normal variety X is a coherent subsheaf $\mathcal{F} \subset T_X$ such that

(1) $\mathcal{F}$ is saturated, i.e. $T_X/\mathcal{F}$ is torsion free, and
(2) $\mathcal{F}$ is closed under Lie bracket.

The rank of $\mathcal{F}$ is its rank as a sheaf. Its co-rank is its co-rank as a subsheaf of T_X. The canonical divisor of $\mathcal{F}$ is any Weil divisor $K_{\mathcal{F}}$ such that $\mathcal{O}(K_{\mathcal{F}}) \cong \det(\mathcal{F})$.

In analogy with the classical case of a normal projective variety X where it is expected that the birational geometry X is governed by the positivity properties of the canonical bundle $\mathcal{O}_X(K_X)$, a similar principle holds for foliations. Indeed, for a pair $(X, \mathcal{F})$ of a normal projective X and a foliation $\mathcal{F} \subset T_X$, one would like to construct a birational model X' of X where the geometry of the strict transform of $\mathcal{F}$ becomes particularly simple. As in the classical case, the way to construct such "simpler" birational models of the pair $(X, \mathcal{F})$ should rely on a careful analysis of the positivity properties of the canonical bundle of the foliation $\mathcal{O}_X(K_{\mathcal{F}})$.

In low dimension, the birational classification of foliations has seen many important advancements in recent year:

- for surfaces, there now is a very exhaustive and effective picture of the classification of rank 1 foliations, [6, 2, 7, 8];
- in dimension three, several foundational steps have been established towards a full classification both in the case of rank 1, [1, 5], as well as rank 2 foliations on algebraic threefolds, [10, 4, 11].

In dimension greater than 3, the analogue problem is still quite obscure for several concurring reasons, e.g., the lack of an analogue of resolution of singularities in this context.

The aim of this report is to focus on the new advancements in the birational classification of rank 2 foliations on threefolds.

1. The foliated minimal model program

We will be working with a slightly more comprehensive framework than the one just introduced: namely, we will consider a co-rank 1 foliation $\mathcal{F}$ on a normal algebraic threefold X, and effective divisor Δ on X with coefficients in $\mathbb{R}_{\geq 0}$, such that $K_{\mathcal{F}} + \Delta$ is $\mathbb{R}$-Cartier. The latter condition is necessary in order to be able to discuss intersection numbers for $K_{\mathcal{F}} + \Delta$.

Given a triple $(X, \mathcal{F}, \Delta)$ as above, one would like to construct suitable birational models where the geometry of the triple is as simple as possible. The guiding light in this quest should be the positivity of $K_{\mathcal{F}} + \Delta$ which is measured in terms of the positivity of the intersections of $K_{\mathcal{F}} + \Delta$ with complete curves contained in X.

In analogy with the classical Minimal Model Program (in short, MMP), we expect 2 different types of outcomes. Given a triple $(X, \mathcal{F}, \Delta)$, where X projective, and the singularities of the triple are mild – see below for more on singularities – we would like to algorithmically construct a triple $(X', \mathcal{F}', \Delta')$, and a birational contraction $\pi\colon X \dashrightarrow X'$ such that $\mathcal{F}'$ (resp. Δ') is the strict transform of $\mathcal{F}$ (resp. Δ) under π and:

(1) either $K_{\mathcal{F}'} + \Delta'$ is nef on X', that is, $(K_{\mathcal{F}'} + \Delta') \cdot C \geq 0$ for any complete curve $C \subset X'$; or
(2) X' is covered by rational curves that have negative intersection with $K_{\mathcal{F}'} + \Delta'$.

In contructing X' (and thus, π), we would like to preserve the geometric data encoded in the triple $(X, \mathcal{F}, \Delta)$: in particular, we do not want to alter the linear systems $|m(K_{\mathcal{F}} + \Delta)|$, as those carry many important geometric information about $\mathcal{F}$. In view of this, it follows that case (1) in the above dichotomy should correspond to the case where $K_{\mathcal{F}} + \Delta$ is pseudoeffective, while case (2) corresponds to the non-pseudoeffective case.

A triple $(X', \mathcal{F}', \Delta')$ corresponding to an outcome described in (1) above is called a *minimal model* of $(X, \mathcal{F}, \Delta)$, while it is called a *Mori fibre space* when it corresponds to an outcome described in (2).

The classic starting point in the birational classification of higher dimensional algebraic varieties is the quest for a smooth representative in every birational equivalence class. For foliations, it is not hard to see that this question already has a negative answer for rank 1 foliations on surfaces, cf. [2]. For the purpose of the birational classification, the class of *simple singularities* is the correct analogue of a smooth model in the classical case of the birational classification of algebraic varieties, cf. [4, Definition 2.7] for the precise definition. In dimension 2 and 3, it is proven that for any foliated pair $(X, \mathcal{F})$, where $\mathcal{F}$ has co-rank 1, there always exists a birational model where the strict transform of $\mathcal{F}$ has simple singularities, see [9, 3].

On the other hand, the class of simple singularities is not stable under any meaningful class of birational transformation; hence, it is important to identify a suitable class of singularities that works well for our own purpose. The right class of foliated singularities to consider is that of foliated divisorial log terminal (in short, F-dlt) singularities, an analogue in the category of foliated spaces of that of divisorial log terminal singularities in the MMP, cf. [4, § 3] for the precise definition and more details. F-dlt singularities can be nicely characterized in terms of discrepancy of their log canonical divisor; they contain simple singularities and it can be shown that they are stable under the type of birational transformations that are used in the foliated version of the MMP, cf. next section. Hence, form this point of view, they are the most natural class of singularities that we should consider if we want to work with foliated spaces with simple singularities and classify those.

2. MMP for rank 2 foliations on threefolds

To simplify the notation for triples $(X, \mathcal{F}, \Delta)$, we will omit X and just write $(\mathcal{F}, \Delta)$. We will assume that our pairs $(\mathcal{F}, \Delta)$ have F-dlt singularities and explain how to proceed algorithmically to produce a triple $(X', \mathcal{F}', \Delta')$ which is either a minimal model or a Mori fibre space for $(\mathcal{F}, \Delta)$.

The starting input of our algorithmic construction is an F-dlt pair $(\mathcal{F}, \Delta)$ – for example, we could start with a foliation $\mathcal{F}$ with simple singularities on X.

If $K_{\mathcal{F}} + \Delta$ is nef, then our algorithm can stop immediately, as $(\mathcal{F}, \Delta)$ is its own minimal model. On the other hand, if $K_{\mathcal{F}} + \Delta$ is not nef, the following result provides a way forward in our quest for minimal models or Mori fibre spaces.

Theorem 1. [10, 4] *Let X be a normal projective threefold and let $\mathcal{F}$ be a co-rank one foliation. Suppose that (X, D) is klt for some $D \geq 0$. Let $(\mathcal{F}, \Delta)$ be a F-dlt pair and let H be an ample $\mathbb{Q}$-divisor.*
Then there exist countable many curves $C_1, C_2, \ldots$ such that

$$\overline{NE(X)} = \overline{NE(X)}_{K_{\mathcal{F}}+\Delta \geq 0} + \sum \mathbb{R}_+[C_i].$$

Furthermore, for each i, C_i is a rational curve tangent to $\mathcal{F}$ such that $(K_{\mathcal{F}} + \Delta) \cdot C_i \geq -6$, and if $C \subset X$ is a curve such that $[C] \in \mathbb{R}_+[C_i]$ then C is tangent to $\mathcal{F}$. In particular, there exist only finitely many $(K_{\mathcal{F}} + \Delta + H)$-negative extremal rays.

Given a $K_{\mathcal{F}} + \Delta$-negative extremal ray $R \subset \overline{NE}(X)$, we can look at the set $\mathrm{loc}(R)$ of all points $x \in X$ such that there exists a curve C with $x \in C$ and $[C] \in R$. We have the following three distinct possibilities:

- if $\mathrm{loc}(R) = X$, then [10, Theorem 8.9] implies that R is K_X-negative and so there exists a contraction $f: X \to Y$ with $\dim X > \dim Y$. Thus, X is covered by rational curves that have negative intersection with $K_{\mathcal{F}} + \Delta$, and it is a Mori fibre space; in this case we can stop our algorithm at this stage.
- If $\mathrm{loc}(R) = D$ is a divisor, then it is shown in [10, 4] that D can be contracted by means of a birational contraction $f: X \to Y$. Such a morphism f is called a divisorial contraction; moreover, f preserve the linear systems $|m(K_{\mathcal{F}}+\Delta)| = |m(K_{\mathcal{F}_Y}+\Delta_Y)|$. In this case, we substitute $(X, \mathcal{F}, \Delta)$ with $(Y, \mathcal{F}_Y, \Delta_Y := f_*\Delta)$, where $\mathcal{F}_Y$ is the strict transform of $\mathcal{F}$, and repeat our algorithm starting with this new triple.
- If $\mathrm{loc}(R) = C$ is a curve, then there exists a birational contraction $f: X \to Y$ whose exceptional locus is Y. Such an f is called a flipping contraction. In this case, $f_*(K_{\mathcal{F}}+\Delta)$ ceases to be $\mathbb{R}$-Cartier, as f is a small contraction but $K_{\mathcal{F}} + \Delta$ intersects C negatively. Hence, we cannot just substitute X with Y and $(\mathcal{F}, \Delta)$ with their strict transforms, as we would not be able to measure the positivity of $K_{\mathcal{F}} + \Delta$ anymore on Y.

In all the of cases above, by Theorem 1, the morphism f contracts curves tangent to the foliation, thus, f is equivariant with respect to the foliation.

In the last case above, that of a so-called flipping contraction, can be remedied by means of the so-called flip of $f: X \to Y$. A flip is nothing more than the following diagram of birational maps

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & Y & \end{tikzcd}$$

where $f^+ \colon X^+ \to Y$ is also a birational contraction whose exceptional locus has dimension 1 and $K_{\mathcal{F}^+} + \Delta^+$ is $\mathbb{R}$-Cartier and it has positive intersection with all curves contracted by f^+, where $\mathcal{F}^+$ (resp. Δ^+) is the strict transform of $\mathcal{F}$ (resp. Δ).

Theorem 2. [4] *Let $\mathcal{F}$ be a co-rank one foliation on a $\mathbb{Q}$-factorial projective threefold X. Let $(\mathcal{F}, \Delta)$ be a F-dlt pair on X.*
Let $\phi \colon X \to Y$ be a $(K_{\mathcal{F}} + \Delta)$-flipping contraction. Then the $(K_{\mathcal{F}} + \Delta)$-flip exists.

As $K_{\mathcal{F}^+} + \Delta^+$ is $\mathbb{R}$-Cartier, we can still discuss its intersection properties. Moreover, it is possible to prove that when making a flip we have the equality of linear systems $|m(K_{\mathcal{F}} + \Delta)| = |m(K_{\mathcal{F}^+} + \Delta^+)|$ and so we can substitute $(X, \mathcal{F}, \Delta)$ with $(X^+, \mathcal{F}^+, \Delta^+)$ and restart our analysis as we did above.

Theorem 2 is a delicate and fundamental result which relies on a careful analysis of the singularities of $\mathcal{F}$, together with an ingenious argument based on Artin's approximation theorem that is used to produce algebraic approximations to the (possibly formal/trascendental) separatrices[1] of $\mathcal{F}$ around $\mathrm{loc}(R)$.

As for a divisorial contraction $f \colon X \to Y$, the rank of the Picard group of X is strictly greater than that of Y, it follows that when running our algorithm, we can just produce a finite number of divisorial contractions. The same type of result is not a priori clear for the case of flipping contractions and flips. Using the Special Termination for foliated pairs proved in [4] and extending the Bott connection to the case of foliated pairs with terminal singularities, it has been proven in [11] that there cannot be infinite sequences of flips.

Theorem 3. [11] *Let X be a $\mathbb{Q}$-factorial quasi-projective threefold. Let $(\mathcal{F}, \Delta)$ be an F-dlt pair. Then starting at $(\mathcal{F}, \Delta)$ there is no infinite sequence of flips.*

Thus, all of the results contained in Theorem 1-3 can be summarized in the following final theorem, which can be summarized by saying that "the Minimal Model Programme terminates for co-rank 1 F-dlt foliated pairs on projective threefolds".

Theorem 4 (MMP for rank 2 foliations on threefolds)**.** *Let X be a $\mathbb{Q}$-factorial quasi-projective threefold. Let $(\mathcal{F}, \Delta)$ be an F-dlt pair. Then there exists a $(K_{\mathcal{F}} + \Delta)$-negative birational contraction $\pi \colon X \to X'$ and an F-dlt pair $(\mathcal{F}', \Delta')$ on X' such that:*

(1) *either, $K_{\mathcal{F}'} + \Delta'$ is nef; or,*
(2) *there exists a contraction $f' \colon X' \to Y$, with $\dim X' > \dim Y$ and $K_{\mathcal{F}'} + \Delta'$ is ample along the fibers of f'.*

[1] A separatrix is an invariant hypersurface for the foliation F that contains a singular point of the foliation.

In [11], a large suite of applications of the existence of the MMP is shown in the guise of an analysis of local and global properties of foliations on threefolds. The authors study foliation singularities proving the existence of first integrals for isolated canonical foliation singularities, an extension of Malgrange's theorem to the singular case, and derive a complete classification of terminal foliated threefolds singularities. They show the existence of separatrices for log canonical singularities. They also prove some hyperbolicity properties of foliations, showing that the failure of the canonical bundle to be nef implies the existence of entire holomorphic curves contained in the open strata of a natural stratification of the singular locus of the foliation.

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