FOR TODAY: EXPLAIN THE PROOF OF THE BOUNDEDNESS THM FOR ELLIPTIC CY'S W/ SECTION.

- GEOMETRY OF THE BASE
- BOUNDEDNESS OF THE BASE
- BOUNDEDNESS OF THE WHOLE FIBRATION

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CALABI-YAU: X NORMAL PROJECTIVE

$$K_X \sim_{\mathbb{Q}} 0 \quad (\Longleftrightarrow K_X \equiv 0)$$

$$H^i(X, \mathcal{O}_X) = \begin{cases} \\ 0 & 0 < i < \dim X \end{cases}$$

THEOREM Fix $d \in \mathbb{Z}_{>0}$. LET

$$\mathcal{E}_d^{CY} = \left\{ Y \;\middle|\; \begin{array}{l} \dim Y = d, \quad Y \text{ is CY \& CANONICAL} \\ f : Y \longrightarrow X \quad \text{elliptic fibration} \\ X \text{ is RC}, \quad \exists\, X \dashrightarrow Y \text{ rat'l section to } f \end{array} \right\}$$

rationally connected

$\mathcal{E}_d^{CY}$ IS BOUNDED UP TO FLOPS.

$$\begin{array}{ccc} \mathcal{H} & & \mathcal{H}_t \dashleftarrow Y \\ \downarrow & & \\ T \ni t & & \end{array}$$

FLOP

$$\begin{tikzcd} X \arrow[rr, dashed, "\text{flop}"] \arrow[dr] & & X' \arrow[dl] \\ & Z & \end{tikzcd}$$

$K_X \cdot C = 0 = K_{X'} \cdot C'$

small contraction

THM [Kawamata]

X, X' are terminal & birat'l + K_X & $K_{X'}$ are nef $\Longrightarrow X \xrightarrow{\text{finite comp. of flops}} X'$

: $\rho(X/Z) = 1 = \rho(X^+/Z)$

RATIONALLY CONNECTED VAR'S :

A PROJECTIVE VAR. X IS RATIONALLY CONNECTED IF $\forall p, q \in X$ general, $\exists\, p \in C \ni q$ RAT'L CURVE.

dim $= 1$ $\qquad \mathbb{P}^1$

dim $= 2$ $\qquad$ SMOOTH RC VAR'S $\longleftrightarrow$ RAT'L VAR'S

dim > 2 $\qquad$ RC VAR'S $\supsetneq$ RAT'L VAR'S

$\qquad\qquad\qquad \cup\!\!\restriction \qquad\qquad \not\supseteq$

$\qquad\qquad\qquad$ FANO'S

CRITERION FOR RCness

$\exists\, C \subseteq X$ rat'l s.t.

$$T_X|_C \simeq \bigoplus \mathcal{O}(i) \quad i > 0$$

LET ME FIRST EXPLAIN WHY THE SAME THM HOLDS ALREADY FOR K3'S (OR RATHER K-TRIVIAL VAR'S)

THEOREM LET

$$\mathcal{E}_2^{K\text{-TR, sect.}} = \left\{ Y \;\middle|\; \begin{array}{ll} \dim X = 2 & Y \text{ SMOOTH PROJ.} \\ K_Y \equiv 0 & Y \longrightarrow X \text{ elliptic w/ section } S \end{array} \right\}$$

section

$\Sigma \subset S$

$\downarrow f$ elliptic

$\mathbb{P}^1$ (2)

$p_i \quad i = 1, \dots, 4$ distinct pts on $\mathbb{P}^1$

$$\Gamma = \underline{f^*(\Sigma p_i)} + \Sigma$$

polarization

prime curve $\subseteq S$

$\Gamma \cdot C$

Γ is ample $\qquad \Gamma$ is nef : $\begin{cases} \overset{"}{>} 0 & C \text{ is vertical} \\ \geq 4 & C \text{ horizontal} \neq \Sigma \end{cases}$

$$\Gamma \cdot \Sigma = \left(\Sigma + f^*\left(\sum p_i\right)\right) \cdot \Sigma = \Sigma^2 + 4 = -2+4 > 2$$

$$\underbrace{(K_S + \Sigma) \cdot \Sigma}_{= \Sigma^2} = -2$$

Γ nef
ample

$$\Gamma^2 > 0$$

$$\Gamma^2 = \Sigma^2 + 2\,\Sigma \cdot f^*\left(\sum p_i\right) = -2 + 8 = 6$$

$\Longrightarrow$ Γ nef & big $\quad \mathrm{vol}(\Gamma) = 6$

$\Longrightarrow$ elliptic K3s w/ section are bounded.

LET ME FIRST EXPLAIN WHY THE SAME THM HOLDS ALREADY FOR K3'S (OR RATHER K-TRIVIAL VAR'S) BUT IT WON'T BE TRUE, IF WE DON'T ASSUME THE $\exists$ OF A SECTION.

LET $\quad \mathcal{E}_2^{K\text{-TR.}} = \left\{ Y \;\middle|\; \begin{array}{ll} \dim X = 2 & Y \text{ SMOOTH PROJ.} \\ K_Y \equiv 0 & Y \longrightarrow X \text{ elliptic} \end{array} \right\}_{0}$

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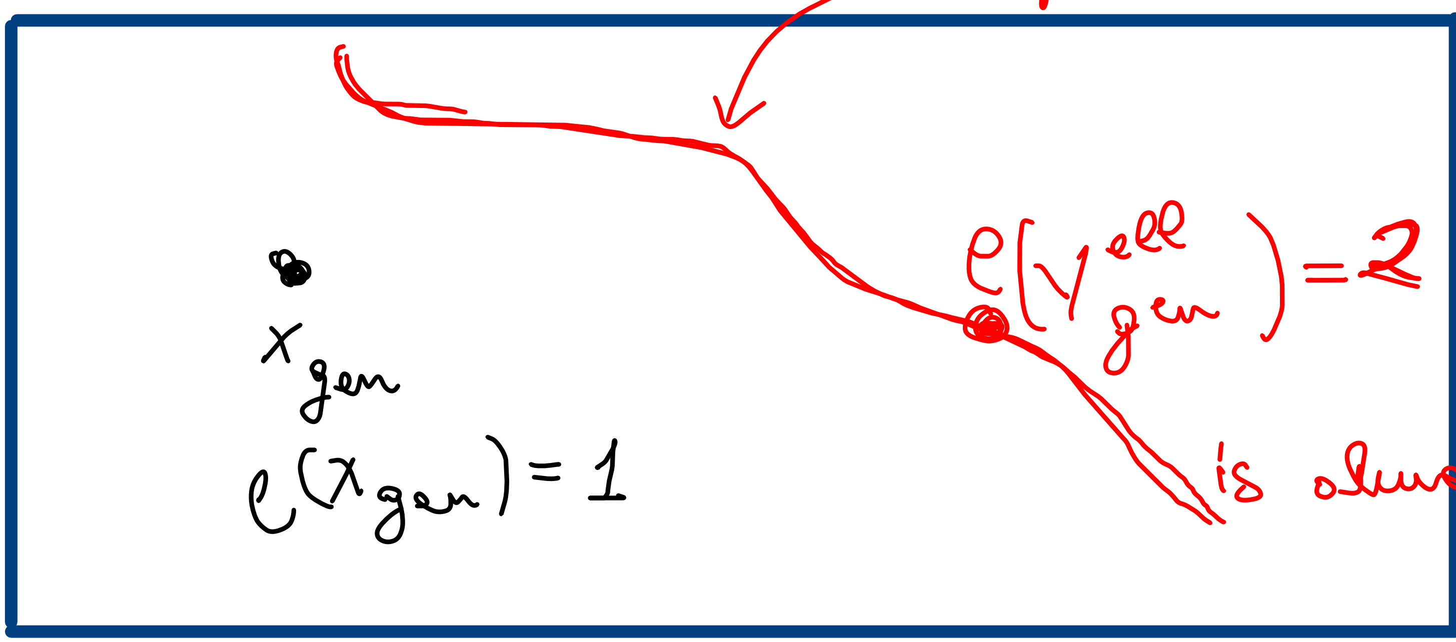
elliptic K3s

$\mathcal{H}_{2d}$



$\mathbb{Z}[H_{2d}]$

$\|$

$\mathrm{Pic}(Y^{ell}_{gen}/\mathbb{P}^1)$

is always a divisor

$2d \geq 6$

ALSO, LET US SHOW THAT IF $Y \xrightarrow{\text{eQ.}} X$ w/ Y ICY + SMOOTH

CANONICAL

$\Rightarrow$ AUTOMATICALLY X IS RATIONALLY CONNECTED.

ALSO, LET US SHOW THAT IF $Y \xrightarrow{\text{eq.}} X$ w/ Y ICY + SMOOTH

$\Rightarrow$ AUTOMATICALLY X IS RATIONALLY CONNECTED.

THEOREM [KOLLÁR-LARSEN] LET Y BE SMOOTH, SIMPLY CONN., $K_Y \sim 0$.

IF $\exists\, g : Y \overset{\text{dominant}}{\underset{\text{rat'l}}{\dashrightarrow}} Z$, $\kappa(Z) \geq 0 \Rightarrow Y \simeq Y_1 \times Y_2$,

$Y_2 \underset{\text{bir.}}{\simeq} Z$,

$$\begin{tikzcd} Y \arrow[d, "pr_2"'] \arrow[dr, dashed, "g"] & \\ Y_2 \arrow[r, dashed, "\text{bir.?}", "\text{isom.}"'] & Z \end{tikzcd}$$

max'lly rat'l connected

MRC fibration : $m : Z \overset{\text{smooth rat'l}}{\underset{\text{quasi-holomorphic}}{\dashrightarrow}} T$

(indet. locus does not dominate T)

T does not contain rat'l curves thru a general point.

over $U \underset{\text{zar. open}}{\subseteq} T$

$m^{-1}(u)$ is RC max'l with the appropriate

ALSO, LET US SHOW THAT IF $Y \xrightarrow{\text{eQ.}} X$ w/ Y ~~K~~ CY + SMOOTH

$\Rightarrow$ AUTOMATICALLY X IS RATIONALLY CONNECTED.

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$h^{\dim Y_{1/2}}(Y, \mathcal{O}_Y) \neq 0$

$$\begin{tikzcd} Y \arrow[d] \\ X \end{tikzcd}$$

X not RC

$$\begin{tikzcd} Y \arrow[d] \arrow[dr, dashed] & \\ X \arrow[r, dashed, "m_X"] & Z \end{tikzcd}$$

$\dim Z > 0$

$\kappa(Z) \geq 0$

$\overset{KL}{\Longrightarrow} Y \simeq Y_1 \times Y_2$

$\dim Y_i > 0$

$K_{Y_1} \sim 0$

$h^{\dim Y_1, 0}(Y_1) = \mathbb{C}$

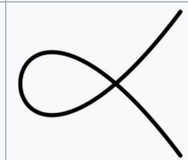
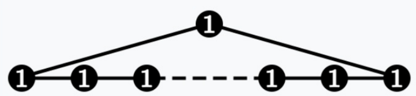
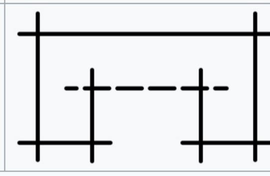
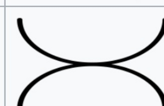
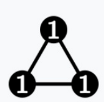
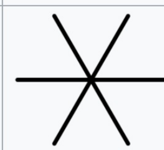
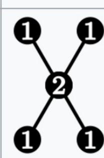
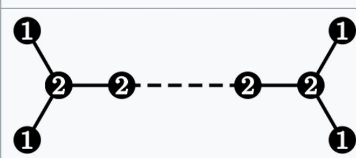
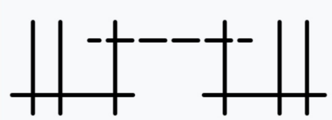
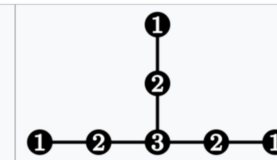
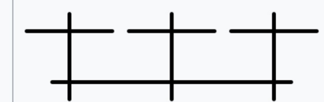
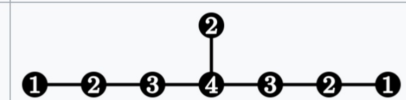
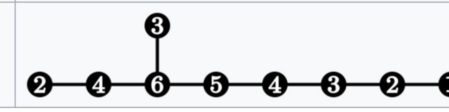
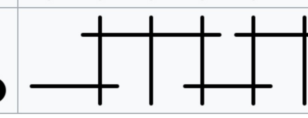
GEOMETRY OF THE BASE

KODAIRA'S FORMULA :

$$\begin{array}{ccl} S & & \text{RELATIVELY MINIMAL} \\ \downarrow g & & \text{ELLIPTIC SURFACE} \\ C & \xrightarrow{\ j\ } & \mathbb{P}^1_j \\ c & \longmapsto & j\text{-inv. of } g^{-1}(c) \end{array}$$

$$K_S \sim g^* L$$

$$\sim g^*\Big(K_C + \underbrace{\sum_{p_i \in C} \mu_i \, p_i}_{\text{depend only on the type of sing's of fibers}} + \frac{1}{12} j^*(\mathcal{O}_{\mathbb{P}^1}(1))\Big)$$

Kodaira	Néron	Components	Intersection matrix	Dynkin diagram	Fiber
I_0	A	1 (elliptic)	0	 lct = 1	
I_1	B_1	1 (with double point)	0	 1	
I_2	B_2	2 (2 distinct intersection points)	affine A_1	 1	
I_v (v≥2)	B_v	v (v distinct intersection points)	affine A_{v-1}	 1	
$_mI_v$ (v≥0, m≥2)		I_v with multiplicity m		$\frac{1}{m}$	
II	C_1	1 (with cusp)	0	 $\frac{5}{6}$	
III	C_2	2 (meet at one point of order 2)	affine A_1	 $\frac{3}{4}$	
IV	C_3	3 (all meet in 1 point)	affine A_2	 $\frac{2}{3}$	
I_0^*	C_4	5	affine D_4	 $\frac{1}{2}$	
I_v^* (v≥1)	$C_{5,v}$	5+v	affine D_{4+v}	 $\frac{1}{2}$	
IV^*	C_6	7 $\frac{1}{3}$	affine E_6		
III^*	C_7	8 $\frac{1}{4}$	affine E_7		
II^*	C_8	9 $\frac{1}{6}$	affine E_8		

$\mu_i = 1 -$

CANONICAL BUNDLE FORMULA :

$$\begin{array}{c} (Y,B) \\ \downarrow \text{fibration} \\ X \end{array} \quad \text{log pair} \quad \text{LC/Klt} \quad K_Y + B \sim_{\mathbb{Q}} f^* L$$

$$K_Y + B \sim f^*\big(K_X + \underbrace{\overbrace{\Gamma_X}^{\text{boundary divisor}}}_{\text{effective canonically det.}} + \underbrace{M_X}_{\text{moduli divisor}}\big)$$

effective canonically det. by looking at lct of fibers/codim 1 pt of X

moduli divisor is canonically det linear system

$M_X \dashleftarrow \mathcal{L}$ modular polarization

$X \dashrightarrow \mathfrak{M}_F$

$$\begin{tikzcd} Y \arrow[d, "f"', "\text{elliptic CY}"] \\ X \\ X' \arrow[u, "r"', "\text{resolution}"] \arrow[r, "``j"'] & \mathbb{P}^1_j \end{tikzcd}$$

$$0 \sim_{\mathbb{Q}} K_Y \sim f^*(K_X + \Gamma_X + M_X)$$

coeff's of this divisor are the same as before

$$M_X = \frac{1}{12} r_*\left(j^* \mathcal{O}(1)\right)$$

$D_X \in |r_* j^* \mathcal{O}(1)|$ s.t. $\overbrace{(X, \underbrace{\Gamma_X + \frac{1}{12} D_X})}^{\text{LOG CY pair}}$ klt

$\text{coeffs}\left(\Gamma_X + \frac{1}{12} D_X\right) \in \left\{\overset{1-\frac{1}{m}}{\text{Kodaira fibers coeff's}}\right\} \cup \left\{\frac{1}{12}\right\}$ $K_X + \Gamma_X + \frac{1}{12} D_X \sim_{\mathbb{Q}} 0$

BOUNDEDNESS OF THE BASE

QUESTION ARE LOG CY PAIRS BOUNDED IN ANY GIVEN FIXED DIMENSION $d \in \mathbb{Z}_{>0}$? NO !!

$$\{(X,B) \text{ LOG}^{\text{klt}} \text{ CY}\}$$

$$\not\supseteq \qquad\qquad\qquad \supseteq$$

$$\{\text{Fano var's}\} \qquad\qquad \{\text{K-TRIVIAL VAR'S}\}$$

$$(\overset{\Downarrow}{X}, \underset{\Pi}{B}) \qquad\qquad B = 0$$

$$|-K_X|_{\mathbb{Q}}$$

CONJECTURE [SHOKUROV] Fix $d \in \mathbb{Z}_{>0}$, $\varepsilon \in \mathbb{R}_{>0}$.

$$\left\{ X \;\middle|\; \begin{array}{l} \dim X = d, \ \exists B \text{ on } X \\ \text{s.t. } (X,B) \ \varepsilon\text{-klt}, \ X \text{ is RC}, \\ \qquad K_X + B \equiv 0 \end{array} \right\}$$

This collection is bounded.

THEOREM [BIRKAR] The conjecture holds up to flops.

BOUNDEDNESS

DEFINITION LET $\mathfrak{D}$ BE A COLLECTION OF LOG PAIRS.

WE SAY THAT $\mathfrak{D}$ IS BIRATIONALLY LOG BOUNDED

IF THERE EXISTS

PROJECTIVE MORPHISMS OF SCHEMES OF FINITE TYPE

$$\begin{tikzcd} \mathcal{X} \arrow[d, "h"] & \supseteq & \mathcal{E} \arrow[dll, "h|_{\mathcal{E}}"] \\ T & & \end{tikzcd}$$

SUCH THAT $\forall (X,B) \in \mathfrak{D}$,

$\exists\, t \in T$ S.T.

$X \overset{\psi}{\dashrightarrow} \mathcal{X}_t$ BIRAT'L MAP

&

$$\mathcal{E} \supseteq \mathrm{Supp}(\psi_* B) + \mathrm{Exc}(\psi^{-1})$$

TOOLS FROM THE MMP

AS WE MENTIONED YESTERDAY, PROVING BOUNDEDNESS IS A DIFFICULT TASK IF WE TRY TO ACHIEVE IT BY PRODUCING A VERY AMPLE DIVISOR WITH BOUNDED VOLUME.

TOOLS FROM THE MMP

AS WE MENTIONED YESTERDAY, PROVING BOUNDEDNESS IS A DIFFICULT TASK IF WE TRY TO ACHIEVE IT BY PRODUCING A VERY AMPLE DIVISOR WITH BOUNDED VOLUME.

ON THE OTHER HAND, PRODUCING BIRATIONAL LINEAR SYSTEMS IS A MUCH EASIER TASK – WHICH IS WHAT WE NEED TO PROVE BIRATIONAL BOUNDEDNESS.

EXAMPLE STATEMENT

THEOREM [H-M-X] FIX $d \in \mathbb{Z}_{>0}$, $V \in \mathbb{R}_{>0}$, $I \subset [0,1]$ DCC.

LET $\mathcal{G}$ BE A COLLECTION OF LOG PAIRS (X, Δ) S.T.

(i) X IS PROJ. OF $\dim = d$, COEFF'S $\Delta \in I$,

(ii) $K_X + \Delta$ IS BIG,

(iii) $\mathrm{vol}(K_X + \Delta) \leq V$.

THEN $\mathcal{G}$ IS LOG BIRATIONALLY BOUNDED.

THEOREM [H-M-X] Fix $d \in \mathbb{Z}_{>0}$, $V \in \mathbb{R}_{>0}$, $I \subset [0,1]$ DCC.

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THEN $\mathcal{G}$ is LOG BIRATIONALLY BOUNDED.

THEOREM [H-M-X] Fix $d \in \mathbb{Z}_{>0}$, $\delta, \varepsilon \in \mathbb{R}_{>0}$.

LET $\mathfrak{D}$ BE A COLLECTION OF LOG PAIRS (X, B) S.T.

1. X is PROJ. OF $\dim = d$,
2. $K_X + B$ IS AMPLE,
3. COEFFS OF B are $\geq \delta$
4. $a(X, B) \geq \varepsilon$.

IF $\mathfrak{D}$ is LOG BIRATIONALLY BOUNDED $\Longrightarrow$ $\mathfrak{D}$ IS LOG BOUNDED.

THEOREM Fix $d \in \mathbb{Z}_{>0}$. LET

$$\mathcal{E}_d^{CY} = \left\{ Y \;\middle|\; \begin{array}{l} \dim Y = d, \quad Y \text{ is CY \& CANONICAL} \\ f : Y \longrightarrow X \quad \text{elliptic fibration} \\ X \text{ is RC}, \quad \exists\, X \dashrightarrow Y \text{ rat'l section to } f \end{array} \right\}$$

rationally connected

$\mathcal{E}_d^{CY}$ IS BOUNDED UP TO FLOPS.

$$\begin{tikzcd} \mathcal{H} \arrow[d] & \mathcal{H}_t & Y \arrow[l, dashed] \\ T \ni t \end{tikzcd}$$

FLOP

$$\begin{tikzcd} X \arrow[rr, dashed, "\text{flop}"] \arrow[dr] & & X' \arrow[dl] \\ & Z & \end{tikzcd}$$

$K_X \cdot C = 0 = K_{X'} \cdot C$

Small contraction

THM [Kawamata]

X, X' are terminal & birat'l + K_X & $K_{X'}$ are nef $\Longrightarrow X \xrightarrow{\text{finite comp. of flops}} X'$

: $\rho(X/Z) = 1 = \rho(X^+/Z)$

GLOBAL ACC

Fix I DCC set, $d \in \mathbb{Z}_{>0}$.

$I = \left\{ \frac{m-1}{m} \;\middle|\; m \in \mathbb{Z}_{>0} \right\} \cap \mathbb{Q}$

$I \cup$ finite set

I $\cup$ finite

$\exists \; I_0 = I_0(d, I)$ s.t. $\forall (X, B)$ LC log CY pair

s.t. $\dim X = d$ $\quad K_X + B \equiv 0$

$B \in I$

$\Longrightarrow \; B \in I_0$

COR. Klt log CY + DCC coeffs I

$\Longrightarrow \exists \, \varepsilon = \varepsilon(d, I) > 0$ ε-klt log CY

PROOF LET'S FIRST DO THE PROOF IN A VERY SIMPLE CASE.

LET'S ASSUME THAT $Y \xrightarrow{\text{elliptic}} X$ + Σ is A SECTION $X \underset{\sigma}{\overset{f}{\leftrightarrows}} Y$

Smooth CY — bounded

X bounded $\Longrightarrow$ $\exists\, C > 0$ s.t.

$\exists\, H_X$ very ample w/ $H_X^{\dim X} \leqslant C$.

$H_Y = \Sigma + \ell\, \pi^* H_X \qquad \ell = 2 \dim Y + 2$

H_Y nef $\qquad H_Y \equiv K_Y + H_Y \qquad (Y, H_Y)$ is LC

if $\exists\, C$ s.t. $K_Y + H_Y \cdot C < 0$ $\Longrightarrow \begin{cases} C \text{ vertical } > 0 \\ \quad C\ \Sigma \\ C \text{ horizontal} \not\subset \Sigma \quad > 0 \end{cases}$

$\quad\equiv H_Y \cdot C < 0$

$C \subset \Sigma$

$$\underbrace{\Sigma \cdot C}_{\geq} + l f^*(H_x) \cdot C > 0$$

$$\left(K_Y + \Sigma\right)\big|_\Sigma \cdot C$$

$$\| $$

$$\underbrace{K_\Sigma \cdot C}_{\wedge\, 0} + \underbrace{l\, H_x \cdot C}_{\vee\, 0}$$

$l = 2 \dim X + 2 > 2 \dim \Sigma$

$$\underbrace{-2\dim\Sigma} \leq K_\Sigma \cdot C < 0$$

THEOREM [KOLLÁR, MATSUSAKA, DEMAILLY, SIU]

LET X BE A SMOOTH K-TRIVIAL VAR.

LET H BE AN AMPLE CARTIER DIVISOR ON X.

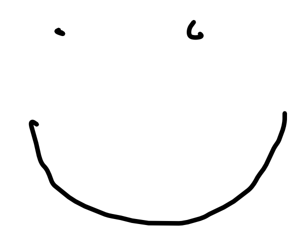
THEN, $\exists\, m = m(\dim X)$ s.t. $|mA|$ IS VERY AMPLE.

$$\mathrm{Vol}(H_Y) = (\Sigma + H_X)^{\dim Y = d}$$

$$= \underbrace{H_X^{\dim Y}}_{= 0} + \Sigma \cdot \underbrace{\left(\sum_{i=0}^{d-1} \binom{d}{i+1} \Sigma^{i} H_X^{d-i-1} \right)}$$

$$\left(\underline{\qquad\qquad\qquad} \right)\Big|_{\Sigma} \qquad \Sigma|_{\Sigma} \equiv K_{\Sigma} \simeq X$$

$$\underset{0}{\overset{}{C' \,\vee}} \;\; \geq \;\; \left(\sum_{i=0}^{d-1} \binom{d}{i+1} \underbrace{K_{\Sigma}^{i} H_X^{d-i-1}}_{\text{finite}} \right)$$



X bounded

$$\begin{tikzcd} \mathcal{H} \subseteq \mathcal{X} \arrow[d] & \supseteq X \arrow[d] \\ T & \ni t \end{tikzcd}$$

sample/T

$$\mathcal{H}^i \cdot K^j_{\mathcal{X}/T} = \text{constant}$$

THEOREM Fix $d \in \mathbb{Z}_{>0}$, $\varepsilon \in \mathbb{R}_{>0}$. LET

$$\mathcal{E}^{K\text{-}TR.}_{d,\varepsilon} = \left\{ Y \;\middle|\; \begin{array}{l} \dim Y = d, \quad Y \text{ is CY}, \\ (Y,0)\ \varepsilon\text{-KLT} \\ f: Y \longrightarrow X \text{ elliptic fibration} \\ X \text{ is RC}, \ \exists\, X \dashrightarrow Y \text{ rat'l section} \end{array} \right\}$$

$\mathcal{E}^{K\text{-}TR}_{d,\varepsilon}$ IS BOUNDED UP TO FLOPS