BOUNDEDNESS

DEFINITION LET $\mathcal{Q}$ BE A COLLECTION OF PROJ. VARIETIES.

WE SAY THAT $\mathcal{Q}$ IS BOUNDED IF THERE EXISTS

A PROJECTIVE MORPHISM OF SCHEMES OF FINITE TYPE

$$\begin{array}{c} \mathcal{X} \\ \Big\downarrow h \\ T \end{array}$$

SUCH THAT $\forall X \in \mathcal{Q}$, $\exists\, t \in T$ SUCH THAT $\mathcal{X}_t := h^{-1}(t)$ IS ISOMORPHIC TO X.

EXAMPLE

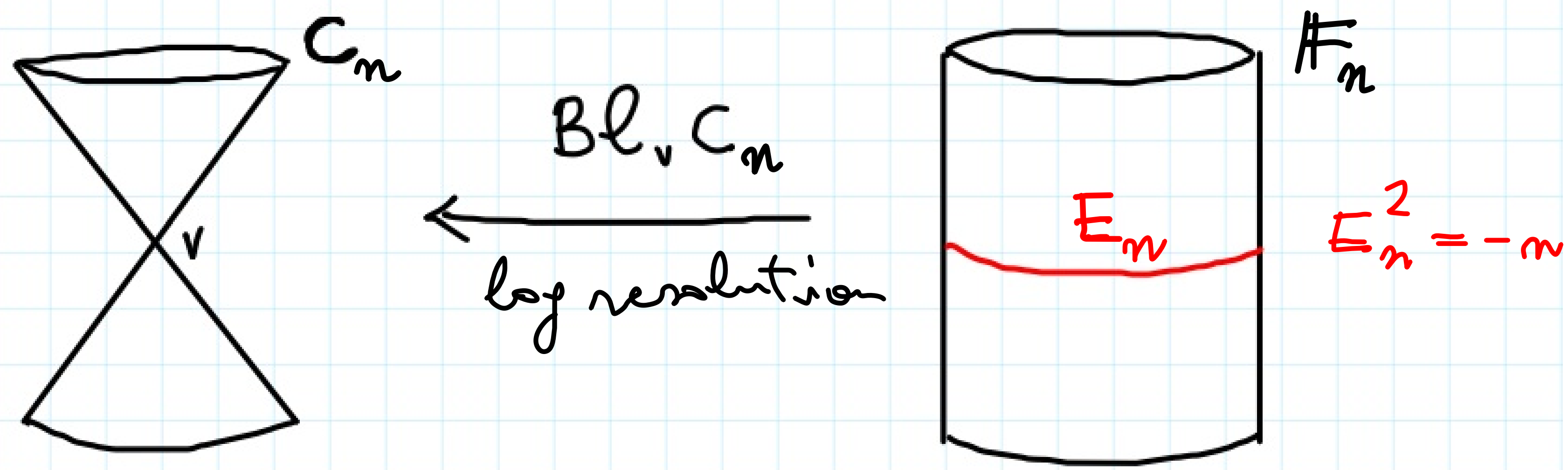
- $\mathcal{DP}^{\text{smooth}} = \{ X \mid X$ IS A del Pezzo SURFACE, SMOOTH $\}$ is BOUNDED

  $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathrm{Bl}_{\leq 8 \text{ pts}} \mathbb{P}^2$

[KMM, Nadel]: $\mathcal{F}_d^{sm} = \{ X \mid X$ Fano smooth terminal, $\dim X = d \}$

IS BOUNDED

EXAMPLE



C_n = CONE OVER RAT'L NORMAL CURVE OF deg n

$(C_n, 0)$ IS KLT AND $a(C_n, 0) = \frac{2}{n}$

$\pi_n^* K_{C_n} = K_{\mathbb{F}_n} + \left(1 - \frac{2}{n}\right) E_n$

- $\mathfrak{D}^{cones} = \{ C_n \mid n \in \mathbb{Z}_{>0} \}$ IS NOT BOUNDED

Let H_n be ample Cartier divisor on C_n.

$$\underbrace{\pi_n^* H_n}_{\substack{\text{Cartier}\\ \text{divisor}\\ \text{big + nef}\\ \text{on } \mathbb{F}_n}} = a \underbrace{E_n}_{\mathrm{exc}(\pi_n)} + b \underbrace{F_n}_{\text{class of a fiber } \mathbb{F}_n \to \mathbb{P}^1}, \quad a, b > 0$$

$$\pi_n^* H_n \cdot E_n = 0 = (a E_n + b F_n) \cdot E_n = -na + b$$

$$\pi_n^* H_n = \ell E_n + \ell n F_n \quad \text{for some } \ell > 0$$

$$(\pi_n^* H_n)^2 = -\ell^2 n + 2n\ell^2 = n\ell^2 \xrightarrow{n \to +\infty} +\infty$$

Let H_n be ample divisor on C_n.

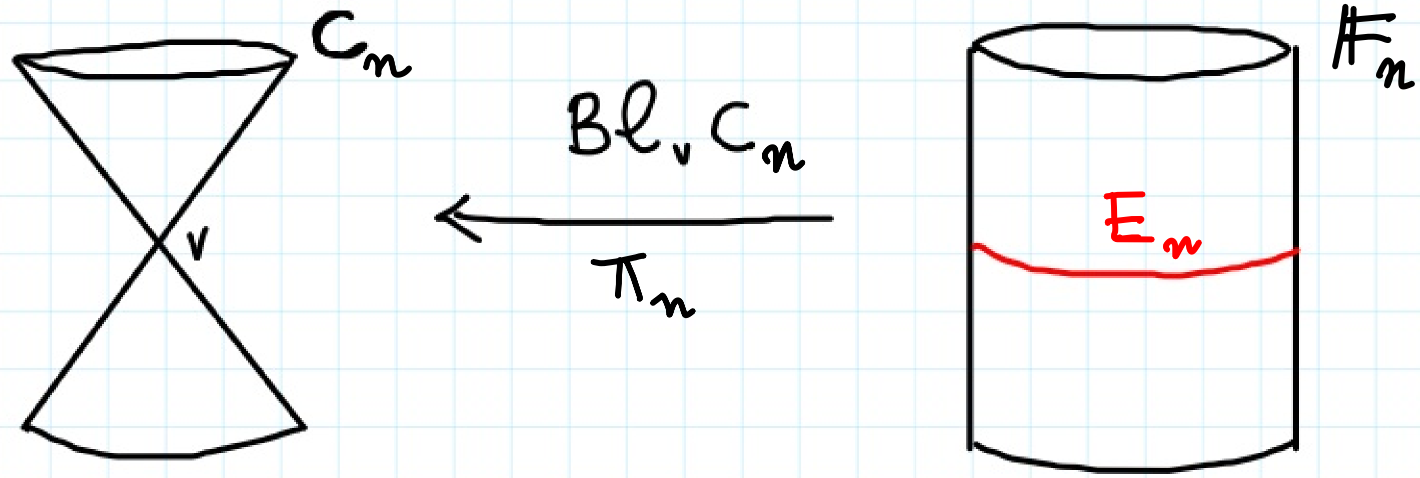
$$\pi_n^* H_n = a E_n + b F_n \quad , \quad a, b > 0$$

Moreover $\pi_n^* H_n \cdot E_n = 0 \implies \underbrace{(a E_n + b F_n) \cdot E_n}_{= -n \cdot a + b} = 0$

Then $\pi^* H_n = l E_n + n l F_n$, for some $l > 0$.

$$(\pi^* H_n)^2 =$$

EXAMPLE



C_n = CONE OVER RAT'L NORMAL CURVE OF deg n

$(C_n, 0)$ IS Klt AND $a(C_n, 0) = \frac{2}{n}$

- $\mathfrak{D}^{\text{cones}}_d = \{ C_n \mid n \leq d \}$ IS BOUNDED FOR FIXED $d \in \mathbb{Z}_{>0}$

(SINCE IT IS A FINITE SET)

BOUNDING $n \iff$ Bounding $\varepsilon \leq a(C_n, 0)$

[Alexeev, 92]: THE COLLECTION OF ε-klt FANO SURFACES IS BOUNDED, FOR ANY FIXED $\varepsilon \in \mathbb{R}_{>0}$.

BOUNDEDNESS

DEFINITION LET $\mathfrak{D}$ BE A COLLECTION OF PROJ. VARIETIES.

WE SAY THAT $\mathfrak{D}$ IS **BIRATIONALLY BOUNDED**

IF THERE EXISTS

A PROJECTIVE MORPHISM OF SCHEMES OF FINITE TYPE

$$\begin{tikzcd} \mathcal{X} \arrow[d, "h"] \\ T \end{tikzcd}$$

SUCH THAT $\forall X \in \mathfrak{D}$, $\exists t \in T$ SUCH THAT $\mathcal{X}_t := h^{-1}(t)$ IS BIRATIONALLY ISOMORPHIC TO X

EXAMPLE • Fix $d \in \mathbb{Z}_{> \overset{1}{\cancel{0}}}$. LET

$$\mathcal{R}_d = \left\{ X \;\middle|\; \begin{array}{l} X \text{ IS SMOOTH PROJECTIVE, } X \text{ is RATIONAL,} \\ \dim X = d \end{array} \right\}$$

$\mathcal{R}_d$ is BIRATIONALLY BOUNDED BUT NOT BOUNDED.

$X \in \mathcal{R}_d$ is birat. isom to $\mathbb{P}^d$

$$\begin{array}{c} \mathbb{P}^d \\ \downarrow \\ pt. \end{array}$$

LEMMA Fix $d \in \mathbb{Z}_{>0}$. LET $\mathcal{D}$ BE A COLLECTION OF PROJ. VAR'S OF $\dim = d$.

$\mathcal{D}$ IS BIRAT. BOUNDED $\Longleftrightarrow$ $\exists\, C = C(\mathcal{D}) \in \mathbb{Z}_{>0}$ S.T. $\forall X \in \mathcal{D}$, $\exists\, G_X$ BIG WEIL DIVISOR ON X

SUCH THAT

$$\varphi_{|G_X|} : X \dashrightarrow \mathbb{P}^{h^0(G_X)-1}$$

IS BIRATIONAL ONTO ITS IMAGE

AND $\mathrm{vol}(G_X) \leq C$.

EXAMPLE • FIX $d \in \mathbb{Z}_{>0}$. LET

$$\mathcal{R}_d = \{ X \mid X \text{ IS SMOOTH PROJECTIVE, } X \text{ IS RATIONAL, } \dim X = d \}$$

$\mathcal{R}_d$ IS BIRATIONALLY BOUNDED BUT NOT BOUNDED.

• HARD: [J. LIU, '05]

$$\mathcal{F}_3^{klt} := \{ Y \mid Y \text{ IS A KLT FANO 3-FOLD} \}$$ IS NOT BIRATIONALLY BOUNDED!

$$\mathcal{F}_2^{klt} \subseteq \mathcal{R}_2$$

BIR.
BOUNDED

BOUNDEDNESS

DEFINITION LET $\mathfrak{D}$ BE A COLLECTION OF LOG PAIRS.

WE SAY THAT $\mathfrak{D}$ IS LOG BOUNDED

IF THERE EXISTS

PROJECTIVE MORPHISMS OF SCHEMES OF FINITE TYPE

$$\begin{tikzcd} \mathcal{X} \supseteq \mathcal{E} \arrow[d, "h"'] \arrow[d, shift left=4, "h|_{\mathcal{E}}"] \\ T \end{tikzcd}$$

SUCH THAT $\forall (X, B) \in \mathfrak{D}$, $\exists\, t \in T$ S.T. $X \xrightarrow{\pi} \mathcal{X}_t$ ISOM. $+$ $\pi(\mathrm{Supp}(B)) = \mathcal{E}_t$

$(\mathcal{X}_t, \mathcal{E}_t)$

variety

reduced divisor

THEOREM [ALEXEEV IN $\dim = 2$, HACON-M^{c}KERNAN-XU]

descending chain condition

FIX $d \in \mathbb{Z}_{>0}$, $v \in \mathbb{R}_{>0}$, $I \subseteq [0,1]$ A DCC SET. LET

[I does not contain strictly decreasing seq.]

$$\mathcal{L}cm_{d,v,I} := \left\{ (X,B) \;\middle|\; \begin{array}{l} \dim X = d, \ (X,B) \text{ is (s)LC}, \\ K_X + B \text{ AMPLE}, \ (K_X+B)^d = v, \\ \text{coeff's of } B \in I \end{array} \right\}.$$

THE COLLECTION $\mathcal{L}cm_{d,v,I}$ IS LOG BOUNDED.

[KOLLÁR, KOVÁCS-PATAKFALVI, ...]: THEOREM

$\Downarrow$

$\exists$ Moduli spaces for log canonical models

$\overline{\mathcal{M}}^{KSBA}_{d,v,I}$ parametrizing isomorphism classes of pair (X,B) as above

THEOREM [ALEXEEV IN $\dim = 2$, BIRKAR]

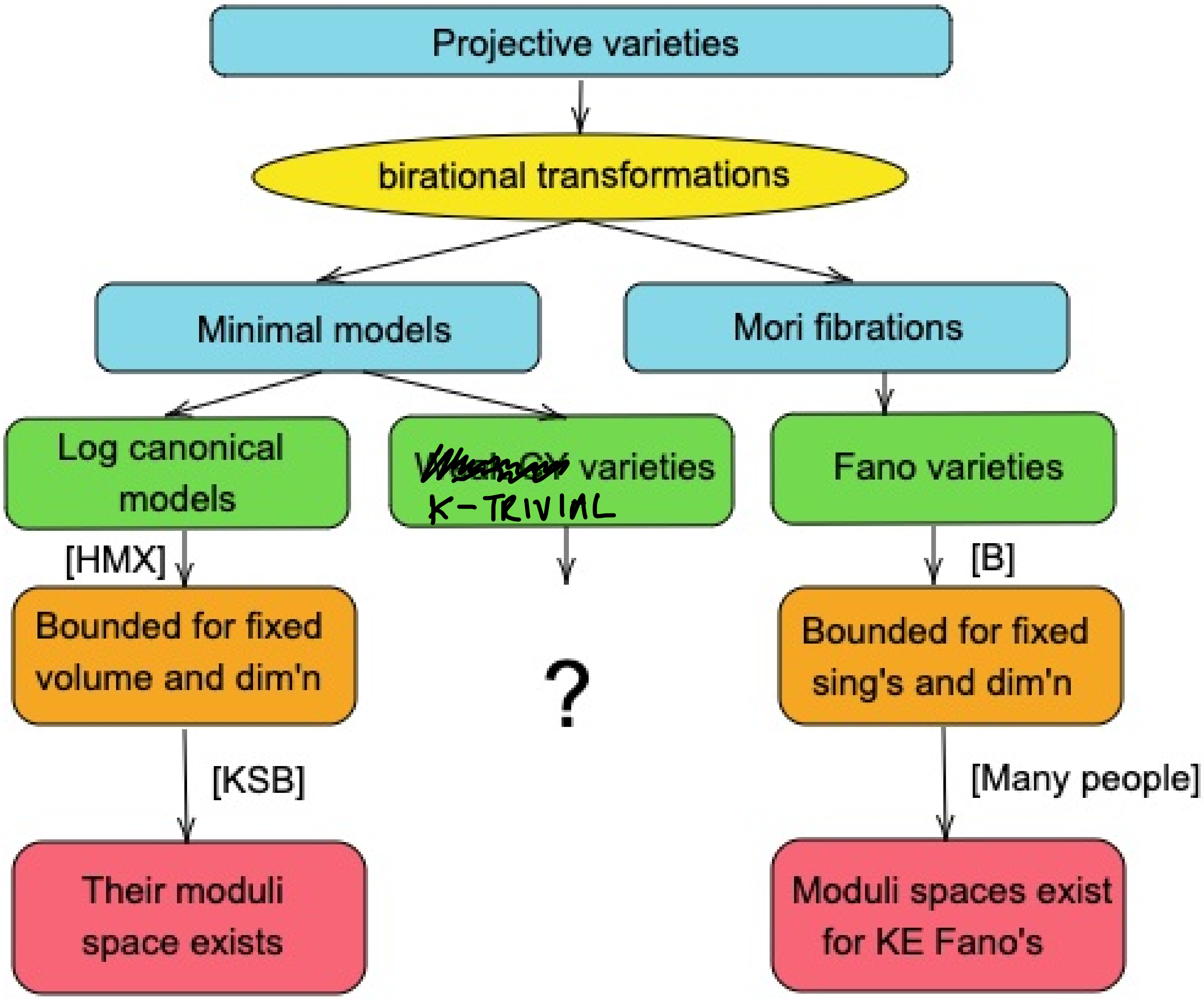
Fix $d \in \mathbb{Z}_{>0}$, $\varepsilon \in \mathbb{R}_{>0}$. LET

$$\mathcal{F}_{d,\varepsilon} := \left\{ X \;\middle|\; \begin{array}{l} \dim X = d, \ \exists B \text{ on } X \text{ s.t.} \\ a(X,B) \geq \varepsilon, \ -(K_X + B) \text{ ample} \end{array} \right\}.$$

THE COLLECTION $\mathcal{F}_{d,\varepsilon}$ is BOUNDED.

[Jiang, Xu – Liu – Blum – Zhuang – HL – ...,]

$\exists$ moduli spaces for K-stable Fano varieties


Projective varieties
birational transformations
Minimal models
Mori fibrations
Log canonical models
varieties
K-TRIVIAL
Fano varieties
[HMX]
[B]
Bounded for fixed volume and dim'n
?
Bounded for fixed sing's and dim'n
[KSB]
[Many people]
Their moduli space exists
Moduli spaces exist for KE Fano's

K-TRIVIAL VAR'S & BOUNDEDNESS

THEOREM [BEAUVILLE - BOGOMOLOV] LET X BE A SMOOTH PROJECTIVE K-TRIVIAL VARIETY. THEN,

$$\exists \; X \xleftarrow{\text{ÉTALE}} X' = \underset{\text{ABELIAN}}{A} \times \prod_{i=1}^{r} \underset{\substack{\text{IRREDUC.}\\ \text{CY}}}{C_i} \times \prod_{j=1}^{s} \underset{\substack{\text{IRRED.}\\ \text{HOL. SYMP.}}}{H_s}$$

$$K \equiv 0 \Rightarrow K \sim_{\mathbb{Q}} 0$$

1^{st} APPROX. ! UNDERSTAND BOUNDEDNESS FOR ABEL., ICY, IHS.

THEOREM [KOLLÁR, MATSUSAKA, DEMAILLY, SIU]

LET X BE A SMOOTH ~~K-TRIVIAL~~ $K_X \sim 0$ VAR.

LET H BE AN AMPLE CARTIER DIVISOR ON X.

THEN, $\exists\, m = m(\dim X)$ s.t. $|mH|$ is VERY AMPLE.

THM $\Longrightarrow$ $\mathfrak{X}_{d,v} = \{ X \mid \dim X = d,\ K_X \overset{\sim}{\not\equiv} 0,\ \exists\, H$ ample Cartier with $H^d \leq v \}$

is bounded.

THEOREM [KOLLÁR, MATSUSAKA, DEMAILLY, SIU]

LET X BE A SMOOTH K-TRIVIAL VAR.

LET H BE AN AMPLE CARTIER DIVISOR ON X.

THEN, $\exists\, m = m(\dim X)$ s.t. $|mA|$ is VERY AMPLE.

dim = 1 BOUNDED

Are not bounded

- K3 SURFACES $\forall d \in \mathbb{Z}_{>0}$ $\mathcal{H}_{2d} = \{ X \mid X \text{ is K3 } \exists H \text{ ample primitive w/ } H^2 = 2d \}$

  $X_{gen} \in \mathcal{H}_{2d}$ has $\rho(X_{gen}) = 1$

- ABELIAN VARIETIES come in algebraic ∞-families in each fixed dim > 1

- ~~HK~~ IHS VAR'S ARE UNBOUNDED IN ANY FIXED dim

  $\mathrm{Hilb}^{[n]}(K3)$

CY VAR'S & BOUNDEDNESS

CONJECTURE/QUESTION [REID, YAU] FIX $d \in \mathbb{Z}_{>0}$.

ARE IRREDUCIBLE SMOOTH CY VAR'S OF $\dim = d$ BOUNDED EITHER IN THE ALGEBRAIC OR IN THE TOPOLOGICAL SENSE?

CY VAR'S & BOUNDEDNESS

CONJECTURE/QUESTION [REID, YAU] Fix $d \in \mathbb{Z}_{>0}$.

ARE IRREDUCIBLE SMOOTH CY VAR'S OF $\dim = d$ BOUNDED EITHER IN THE ALGEBRAIC OR IN THE TOPOLOGICAL SENSE?

BUT FINALLY, WE HAVE SOME GOOD NEWS

THEOREM [GROSS]

$$\mathcal{E}CY_3 = \left\{ X \;\middle|\; \begin{array}{l} \dim X = 3, \ X \text{ is } 1CY, \\ X \to Y \text{ elliptic}, \ \cancel{Y \text{ is RAT'L surface}} \end{array} \right\}$$

$\mathcal{E}CY_3$ is birationally bounded

[FILIPAZZI - HACON - S] $\mathcal{E}CY_3$ is BOUNDED.

ELLIPTIC CALABI-YAU VARIETIES:

$$X \xrightarrow{f} Y \qquad X \text{ ICY}, \quad f \text{ connected fibers}, \quad \dim X - \dim Y = 1$$

the general fiber of f is elliptic.

[KOLLÁR]: with some condition $c_2(X) \cdot f^* H \neq 0$ $(>)$
any small deformation of an elliptic X ICY
is still elliptic

THEOREM [BIRKAR-DICERBO-S] Fix $d \in \mathbb{Z}_{>0}$.

$$\mathcal{ECY}_d^{rat} := \left\{ X \;\middle|\; \begin{array}{l} \dim X = d, \quad X \text{ ICY}, \\ X \longrightarrow Y \text{ elliptic} + Y \underset{\text{rat'l section}}{\dashrightarrow} X \end{array} \right\}$$

$\mathcal{ECY}_d^{rat}$ is birationally bounded [bounded up to flops]

$$\begin{tikzcd} \mathcal{X} \arrow[d] \\ T \end{tikzcd}$$

$\forall\, X \in \mathcal{ECY}_d^{rat}$

$\exists\, t \in T \qquad \mathcal{X}_t \overset{\text{birat'l}}{\underset{\text{isom.}}{\dashleftarrow}} X$

isom. in codim 1

$K_{\mathcal{X}_t} \equiv 0$

composition of flops

$$b_2(X^3) \gg 1 \implies \exists \begin{tikzcd} X' \arrow[d, "\text{elliptic}"] & X \arrow[l, dashed, "\sim"'] \\ Y \end{tikzcd}$$

[Wilson]

STRATEGY FOR GROSS'S THM

$$\begin{array}{ccc} X & \text{ICY} & \\ \downarrow & & \\ Y & & \end{array} \rightsquigarrow \begin{array}{ccc} X_J & \text{CY} & \\ \downarrow \uparrow & \text{rat'l section} & \\ Y_J & & \end{array} \rightsquigarrow$$

DIVIDE & CONQUER : Study Y_J str. + Boundedness & study boundedness of X_J using Σ_J

$\rightsquigarrow$ X_J is bounded

$\rightsquigarrow$ Study $Ш_{X/Y_J}$ Tate Shafarevich is finite