THE CONE THEOREM

THEOREM

LET (X,B) BE A LOG CANONICAL PAIR.

WE HAVE THE FOLLOWING DECOMPOSITION

$$\overline{NE}(X) = \overline{NE}(X)_{K_X+B\geq 0} + \overline{NE}(X)_{K_X+B<0} =$$

$$= \overline{NE}(X)_{K_X+B\geq 0} + \sum_{i\in I} \overbrace{\mathbb{R}_+[C_i]}^{R_i \text{ extremal ray}}$$

at most countable set

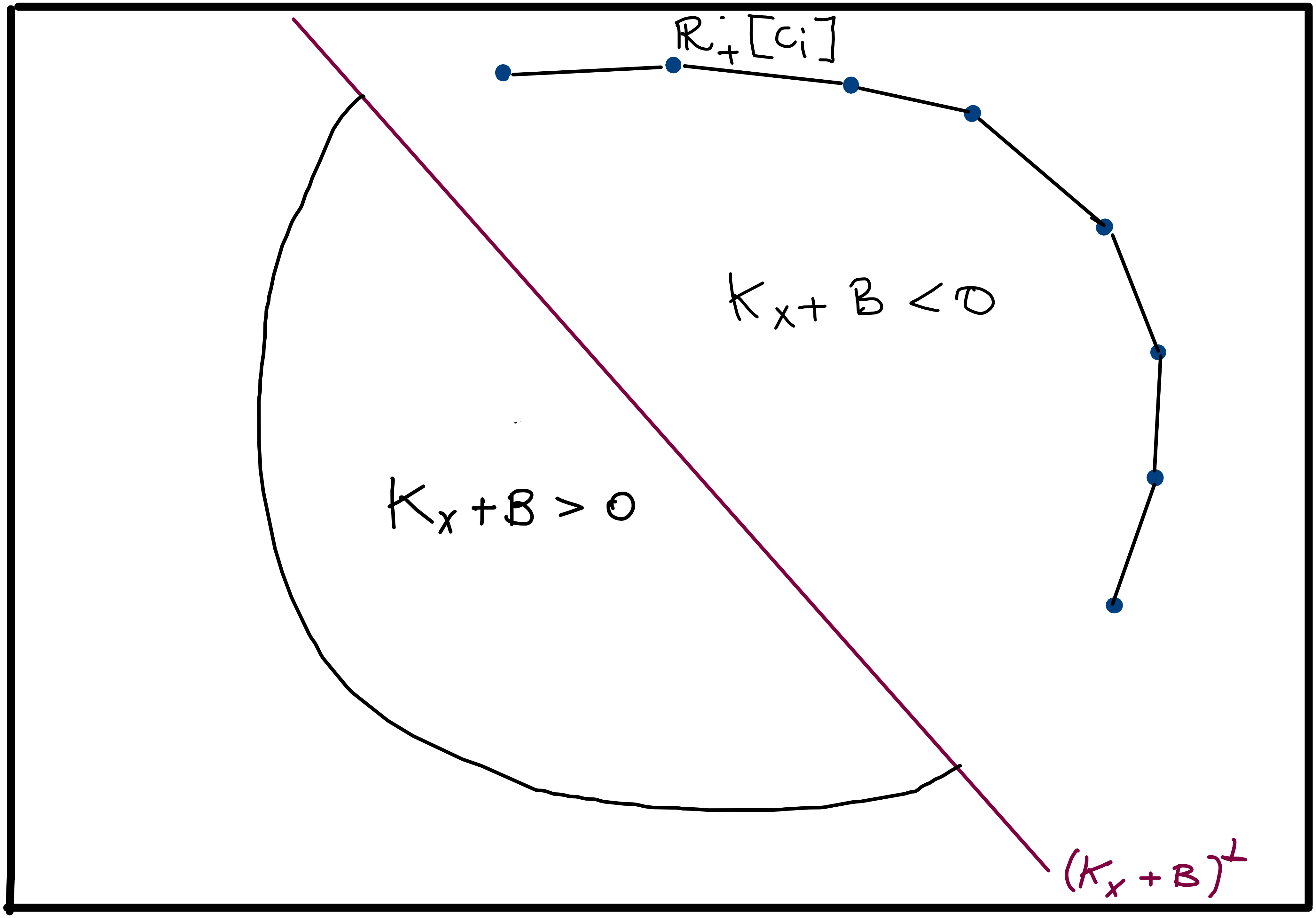
rat'l curves

$-2\dim X \leq (K_X+B)\cdot C_i < 0$

MOREOVER, $\forall i \in I$, $\exists$ $\mathrm{cont}_{R_i} : X \longrightarrow Z_i$ SUCH THAT cont_{R_i} HAS CONNECTED FIBERS & IF $\mathrm{cont}_{R_i}(C) = pt \Rightarrow [C] \in R_i$.

HORIZONTAL SLICE OF $\overline{NE}(X)$

extremal rays $\overset{\text{def}}{\iff}$ $v_1 + v_2 \in R_i^- \Downarrow v_1, v_2 \in R_i^-$



THERE ARE 3 POSSIBLE OUTCOMES OF A CONTRACTION OF AN EXTREMAL RAY:

① DIVISORIAL CONTRACTION

$\mathrm{cont}_{R_i}: X \longrightarrow Z_i$ is birational + $\mathrm{Exc}(\mathrm{cont}_{R_i}) = D \subseteq X$ is a prime divisor.

<u>Ex</u>: $Y \xrightarrow{\mathrm{Bl}_C X} X$

② MORI FIBER SPACE

$\mathrm{cont}_{R_i}: X \longrightarrow Z_i$ is a fibration $-(K_X+B)|_F$ is ample

③ SMALL CONTRACTION

$\mathrm{cont}_{R_i}: X \longrightarrow Z_i$ is birational + $\mathrm{Exc}(\mathrm{cont}_{R_i})$ has codim ≥ 2 in X

START WITH (X, B) LOG CANONICAL/KLT

IS $K_X + B$ NEF ?

YES → STOP MINIMAL MODEL.

NO ↓

START WITH (X, B) LOG CANONICAL/KLT

IS $K_X + B$ NEF ?

$\xrightarrow{\text{YES}}$ STOP (X, B) is A MINIMAL MODEL

$\downarrow$ NO

$\mathrm{Cont}_R : X \longrightarrow Z$

$\xrightarrow{\text{MFS}}$ Stop

$\xrightarrow{\text{SMALL CONTR.}}$

$X \rightsquigarrow X^+$

$B \rightsquigarrow B^+$

restart

$\downarrow$ DIV. CONTR.

continue by substituting

$X \rightsquigarrow Z$ $\quad$ $B \rightsquigarrow \tilde{B}_Z$

EXISTENCE OF FLIPS

Let (X, B) be a LC pair.

ASSUME THAT $K_X + B$ IS NOT NEF & WE CONTRACT A $(K_X + B)$-NEGATIVE EXTREMAL RAY $R \subset \overline{NE}(X)$

EXISTENCE OF FLIPS

Let (X, B) be a LC pair.

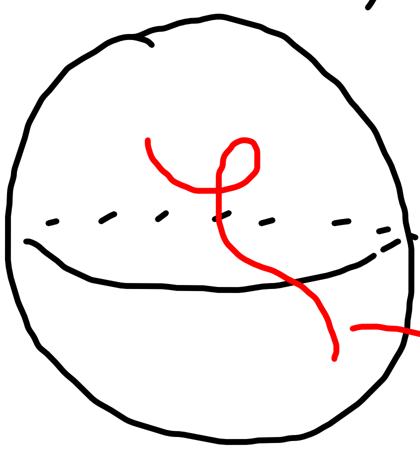
ASSUME THAT $K_X + B$ IS NOT NEF & WE CONTRACT A (K_X+B)- NEGATIVE EXTREMAL RAY $R \subset \overline{NE}(X)$ THROUGH A SMALL CONTRACTION

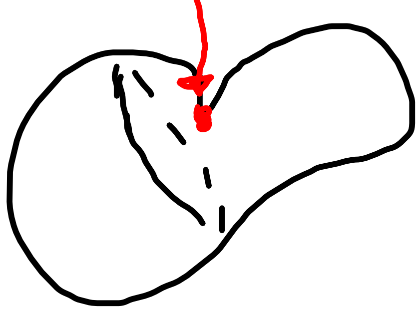
$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr, "\text{birat'l}"'] & & X^+ \arrow[dl, "\text{birat'l morphism small contraction}"] \\ & Z & \end{tikzcd}$$

$K_{X^+} + B^+$ $\mathbb{R}$- Cartier

$K_{X^+} + B^+ \cdot C > 0$

B^+





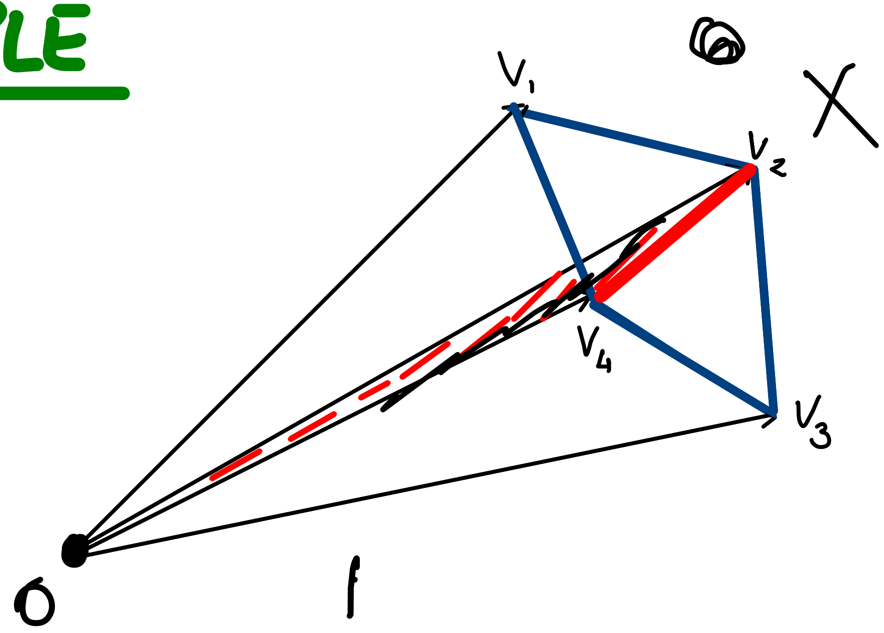
K_Z, $\tilde{B}$

↑ strict transform of B on Z

$K_Z + \tilde{B}$ will never be $\mathbb{R}$- Cartier

EXAMPLE

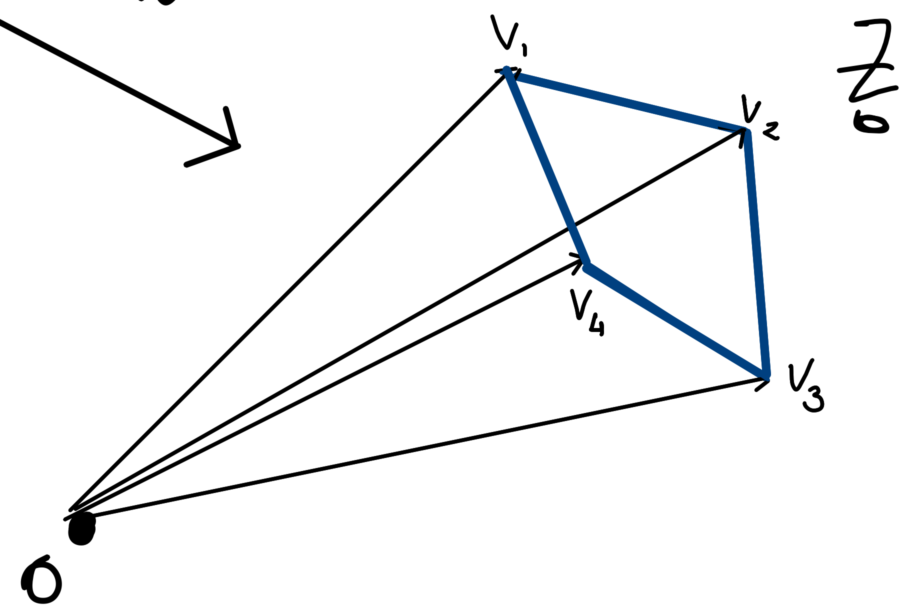
$$\boxed{2v_1 + v_3 = v_2 + v_4}$$



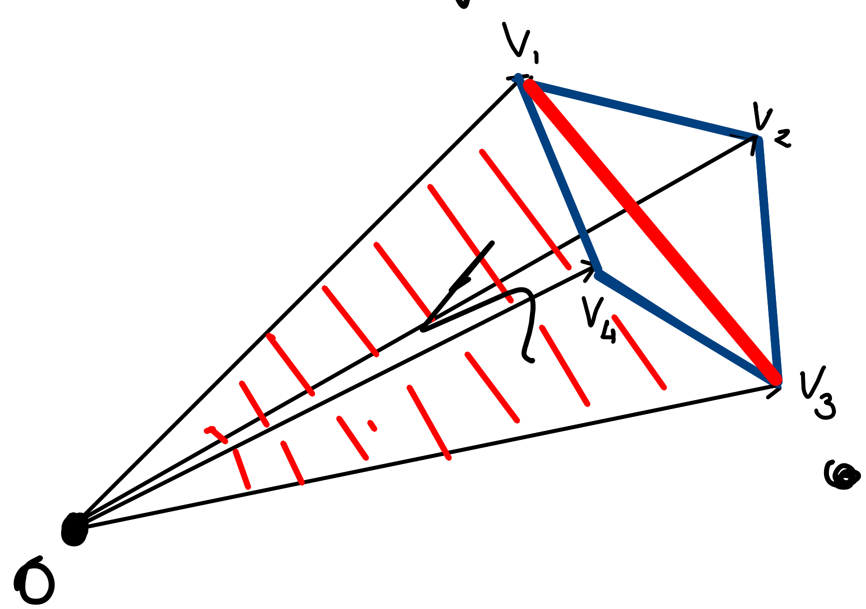
$K_X \cdot R < 0$

$v_i \in \mathbb{Z}^3 \subseteq \mathbb{R}^3$

cont_R



flip of cont_R



X^+

$K_{X^+} \cdot R^+ > 0$

Step 0 *(Initial datum)* Assume that we already constructed a LC pair (X_i, Δ_i) with X_i $\mathbb{Q}$-factorial.

Step 1 *(Preparation)* If $K_{X_i} + \Delta_i$ is nef, go to step 3, case (2). If not, we establish the following results.

(1) (Cone Theorem) $\overline{NE}(X_i) = \overline{NE}(X_i)_{K_{X_i}+\Delta_i \geq 0} + \sum \mathbb{R}_{\geq 0} C_i$.
(2) (Contraction Theorem) Any $K_{X_i} + \Delta_i$-negative extremal ray can be contracted.

Step 2 *(Birational transformations)* If $\mathrm{cont}_{R_i} : X_i \to Y_i$ is birational, then we produce a new pair as follows.

(1) (Divisorial contraction) If cont_{R_i} is a divisorial contraction, then set $X_{i+1} = Y_i$ adn $\Delta_{i+1} = (\mathrm{cont}_{R_i})_* \Delta_i$.

(2) (Flipping contraction) If cont_{R_i} is a flipping contraction, then set $(X_{i+1}, \Delta_{i+1}) = (X_i^+, \Delta_i^+)$, the flip of cont_{R_i}.

In both cases, we produce a LC-pair (X_{i+1}, Δ_{i+1}) with X_{i+1} $\mathbb{Q}$-factorial. Thus, go back to Step 0.

Step 3 *(Final outcome)* We expect that eventually the procedure stops, and we get one of the following two possibilities.

(1) (Fano contraction) If cont_{R_i} is a Fano contraction, then set $X^*, \Delta *) = (X_i, \Delta_i)$.
(2) (Minimal model) If $K_{X_i} + \Delta_i$ is nef then set $X^*, \Delta^*) = (X_i, \Delta_i)$.

TERMINATION Let (X,B) a log canonical pair. Let

$$X =: X_0 \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow \dots \dashrightarrow X_i \dashrightarrow \dots$$

be a sequence of $(K_X + B)$-flips.

Is this a finite sequence?

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be a sequence of (K_X+B)-flips.

Is this a finite sequence?

ABUNDANCE Let (X',B') a log canonical pair.

If $K_{X'}+B'$ is nef, then is it semiample?

EXISTENCE OF GOOD MINIMAL MODELS

THEOREM [BIRKAR - CASCINI - HACON - M^{c}KERNAN]

LET (X, B) BE A KLT PAIR, WHERE X IS $\mathbb{Q}$-FACTORIAL.

ASSUME THAT EITHER $K_X + B$ IS BIG OR NON-PSEFF, OR
B ITSELF IS BIG.

THEN THE MMP (WITH SCALING OF AN AMPLE DIVISOR H on X)
CAN BE RUN & TERMINATES IN FINITE TIME

$$(X, B) =: (X_0, B_0) \dashrightarrow (X_1, B_1) \dashrightarrow \quad \dashrightarrow (X_n, B_n)$$

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AND TERMINATES WITH 1 OF THE 2 FOLLOWING OUTCOMES:

1. MFS $\Updownarrow$ $K_X + B$ is non psef $\qquad$ $X_n \downarrow Z$

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2. GOOD MINIMAL MODEL $X_n \to Y$, $K_{X_n} + B_n$ nef but also semi-ample

$\updownarrow$

$K_X + B$ psef

HENCE, WE CAN CONSIDER THE FOLLOWING 3 CLASSES OF PAIRS:

1. LOG CANONICAL MODELS

   (X, B) LC pair, $K_X + B$ ample

2. K-TRIVIAL PAIRS (or LOG CALABI-YAU pairs)

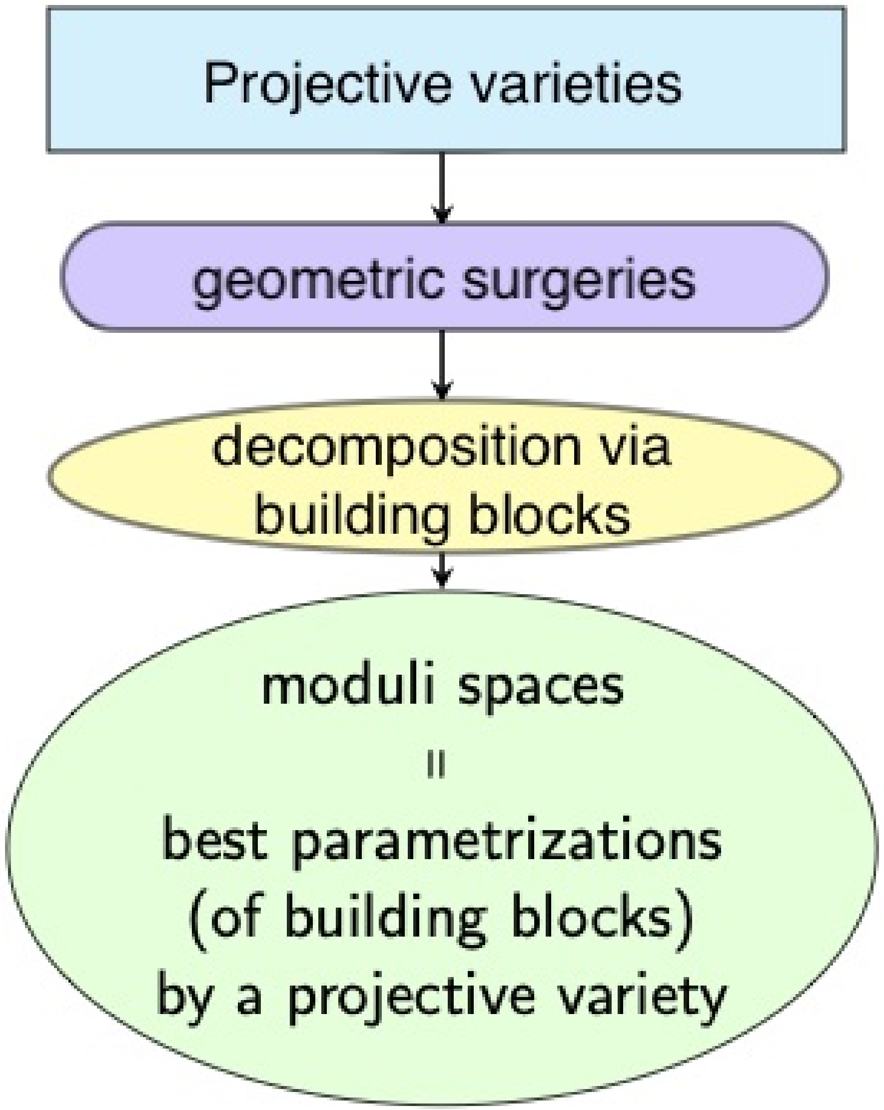
   (X, B) LC pair, $K_X + B \equiv 0$

3. LOG FANO PAIRS (or FANO PAIRS)

   (X, B) LC pair, $-(K_X + B)$ ample

	Log canonical models	Weak CY varieties	Fano varieties
Curvature of KE metric	< 0	= 0	> 0
Rational points	Few [Lang conjecture: {rat'l points} $\subseteq Z \subsetneq X$]	?	Many [Manin conj: |{rat'l pts of height $< B$}| ~ $cB(log\ B)^{b_2-1}$]
Fundamental group	Anything	Virtually abelian	Finite

MINIMAL MODEL PROGRAM: A GENTLE INTRODUCTION



BOUNDEDNESS

BOUNDEDNESS

DEFINITION LET $\mathcal{Q}$ BE A COLLECTION OF PROJ. VARIETIES. WE SAY THAT $\mathcal{Q}$ IS BOUNDED IF THERE EXISTS

A PROJECTIVE MORPHISM OF SCHEMES OF FINITE TYPE

$$\begin{array}{c} \mathcal{X} \\ \downarrow h \\ T \end{array}$$

SUCH THAT $\forall X \in \mathcal{Q}$, $\exists t \in T$ SUCH THAT $\mathcal{X}_t := h^{-1}(t)$ IS ISOMORPHIC TO X.

EXAMPLE LET $\mathcal{C}$ BE THE COLLECTION OF ALL SMOOTH (PROJECTIVE) CURVES ($/\mathbb{C}$).

IS $\mathcal{C}$ BOUNDED ?

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EXAMPLE LET $\mathcal{C}$ BE THE COLLECTION OF ALL SMOOTH (PROJECTIVE) CURVES $(/\mathbb{C})$.

IS $\mathcal{C}$ BOUNDED? NO!

- Ehresmann's theorem is the reason

EXAMPLE LET $\mathcal{C}_g$ BE THE COLLECTION OF ALL SMOOTH (PROJECTIVE) CURVES OF GENUS g.

IS $\mathcal{C}_g$ BOUNDED? YES, FOR ANY FIXED $g \in \mathbb{Z}_{\geq 0}$.

EXAMPLE LET $\mathcal{C}_g$ BE THE COLLECTION OF ALL SMOOTH (PROJECTIVE) CURVES OF GENUS g.

IS $\mathcal{C}_g$ BOUNDED? YES, FOR ANY FIXED $g \in \mathbb{Z}_{\geq 0}$.

$g = 0$ $\quad \mathcal{C}_0 = \{\mathbb{P}^1\} \implies T = \{pt.\}$ SUFFICES

$g = 1$ $\quad \mathcal{C}_1 = \{\text{ELLIPIT CURVES}\} \implies T = \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^2}(3)))$ SUFF.

$g \geq 2$ $\quad \mathcal{M}_g$ $\quad$ BUT ALSO $C \in \mathcal{C}_g \quad C \xrightarrow{|3K_C|} \mathbb{P}^{5g-4}$

$$\implies T = \mathcal{H}ilb(\mathbb{P}^{5g-4})_{3(2g-2)x-g+1}$$

EXAMPLE FIX $n, d, \delta \in \mathbb{Z}_{>0}$. LET

$$\mathcal{D}_{n,d,\delta} := \{ X \subseteq \mathbb{P}^n \mid X \text{ IS A VARIETY w/ } \dim = d, \deg = \delta \}$$

IS $\mathcal{D}_{n,d,\delta}$ BOUNDED?

$d = n-1, \delta$

$$T = \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^n}(\delta)))$$

GENERAL CASE

CHOW VARIETY

LEMMA Fix $d \in \mathbb{Z}_{>0}$. LET $\mathfrak{D}$ BE A COLLECTION OF PROJ. VAR'S OF $\dim = d$.

$\mathfrak{D}$ is BOUNDED $\Longleftrightarrow$ $\exists\, C = C(\mathfrak{D}) \in \mathbb{Z}_{>0}$ s.t. $\forall X \in \mathfrak{D}$, $\exists H_X$ VERY AMPLE CARTIER DIVISOR ON X SUCH THAT $H_X^d \leq C$.

$\Longrightarrow$

$$\begin{array}{ccc} \mathcal{H} \subseteq \mathbb{P}^n \times T & & \mathcal{H}' = \coprod \text{ pullback along strata} \\ \downarrow \text{projective} & & \downarrow \\ T & \longrightarrow & T' = \coprod \text{ strata of } T \end{array}$$

$\Longleftarrow$ $X \in \mathfrak{D}$ $\quad X \xrightarrow{|H_X|} \mathbb{P}^n$ $\quad n \leq \deg X + d \leq C + d$

$X \in \coprod_{n \leq C+d,\ \delta \leq C} \mathrm{Chow}(\mathbb{P}^n)_{\dim = d,\ \deg = \delta}$

EXAMPLE

[KOLLÁR]: IF X $\overset{(X,0)\ \text{LC}}{\text{is LC}}$ AND K_X IS CARTIER + AMPLE $\Longrightarrow$

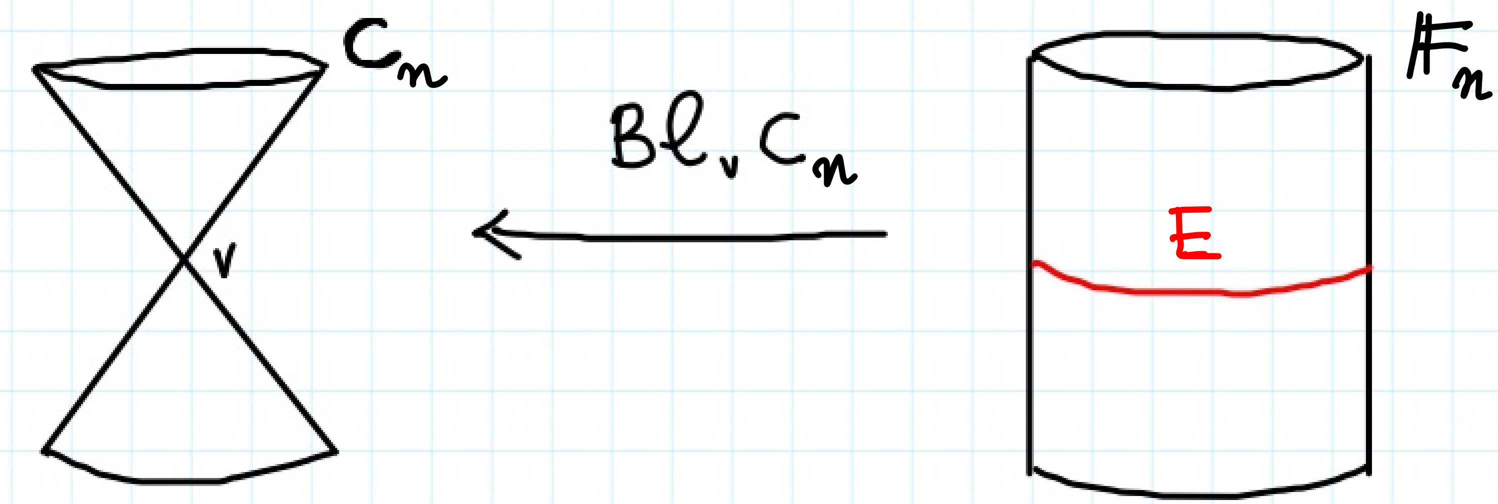
$\exists\ m = m(\dim X)$ S.T. $|mK_X|$ is VERY AMPLE.

$$\mathcal{G}or^v_d = \left\{ X \;\middle|\; \begin{array}{l} X \text{ is LC}, \ K_X \text{ is Cartier + Ample} \\ K_X^{\dim X} = v \end{array} \right\}$$

$\Downarrow$ LEMMA

$\mathcal{G}or^v_d$ is Bounded

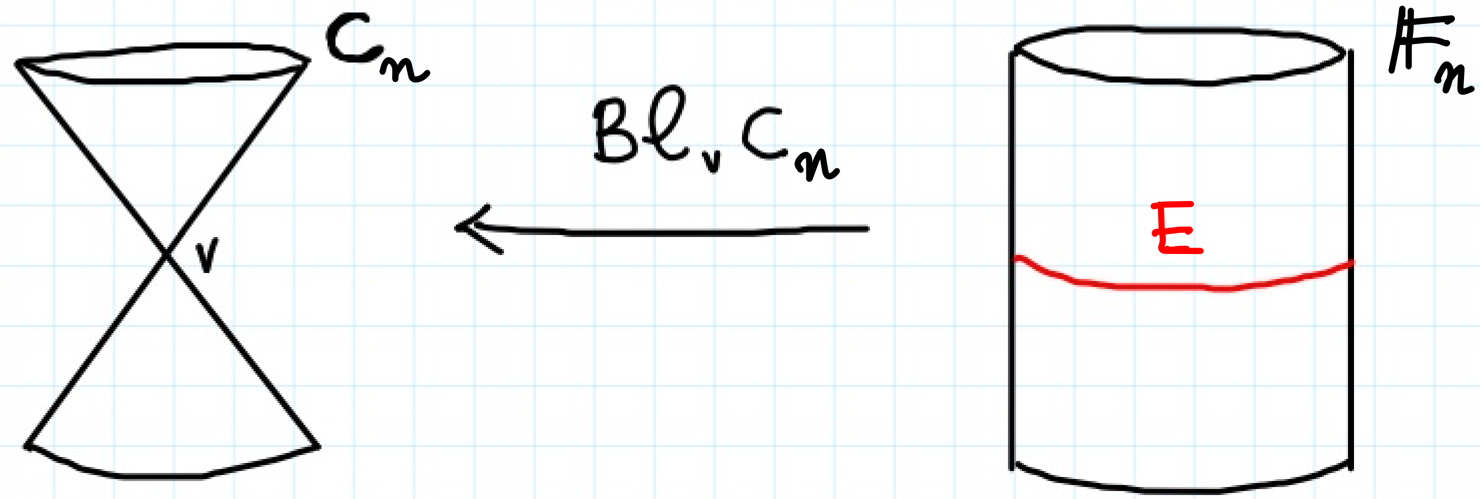
EXAMPLE



C_n = CONE OVER RAT'L NORMAL CURVE OF deg n

$(C_n, 0)$ IS Klt AND $a(C_n, 0) = \frac{2}{n}$

EXAMPLE



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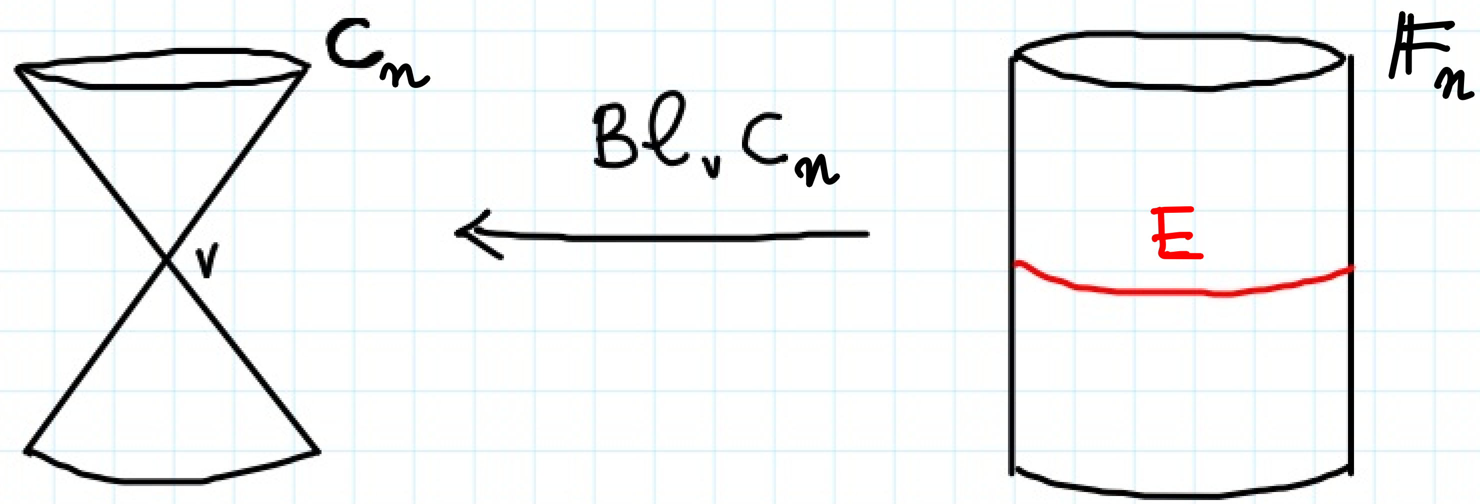

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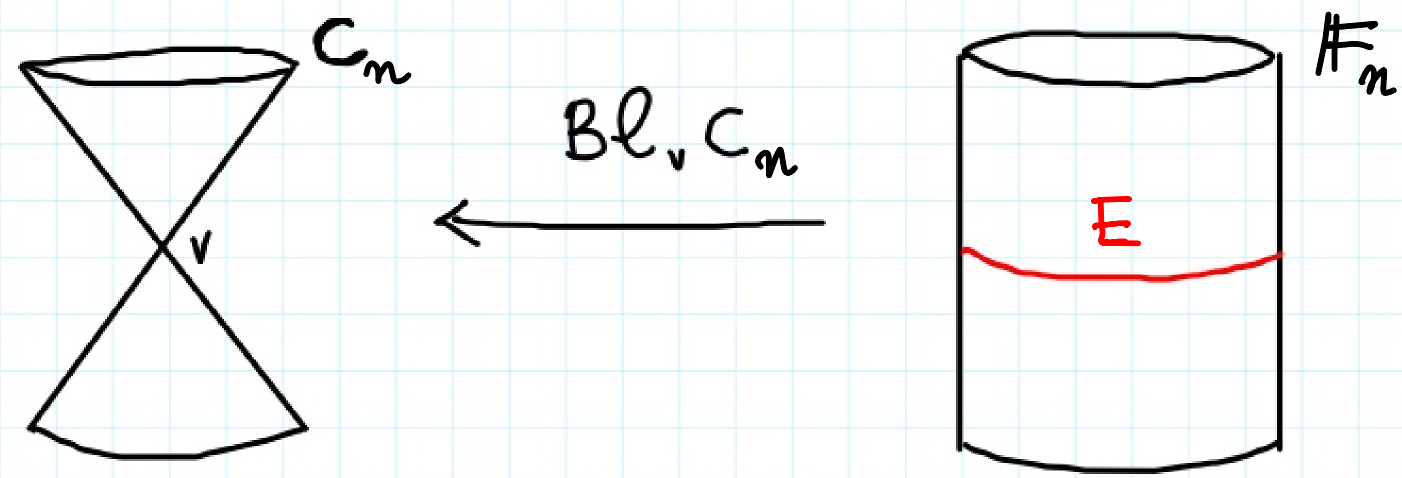
- $\mathcal{DP}^{\text{cones}} = \{ C_n \mid n \in \mathbb{Z}_{>0} \}$ IS NOT BOUNDED

EXAMPLE



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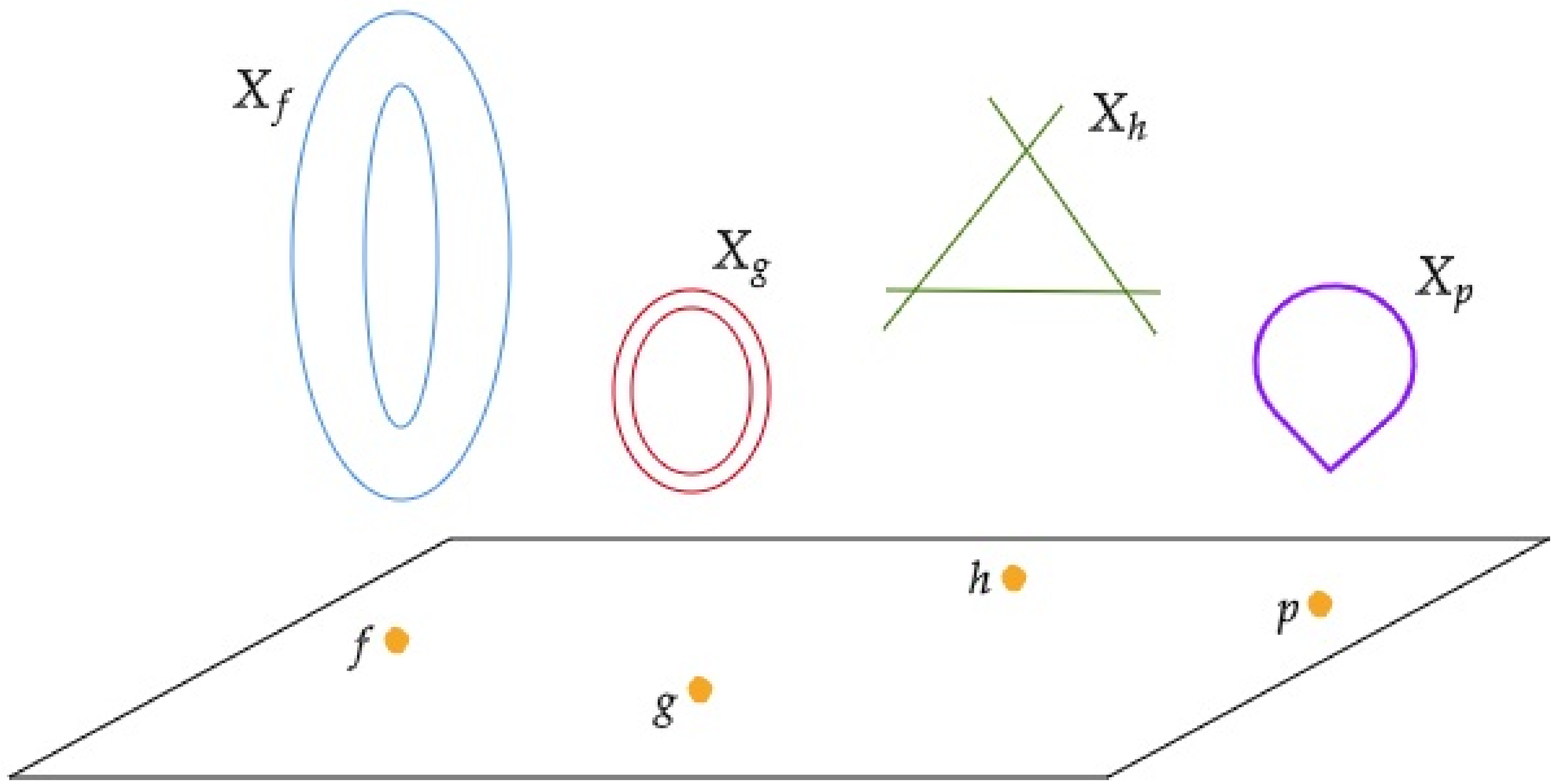
- $\mathfrak{D}\mathfrak{P}^{smooth} = \{ X \mid X$ IS A del Pezzo SURFACE, SMOOTH $\}$ is BOUNDED

- $\mathfrak{D}\mathfrak{P}^{cones}_{d} = \{ C_n \mid n \leq d \}$ IS BOUNDED FOR FIXED $d \in \mathbb{Z}_{>0}$ (SINCE IT IS A FINITE SET)

WHY IS BOUNDEDNESS AN INTERESTING PROPERTY?
IF A COLLECTION $\mathfrak{D}$ OF VARIETIES IS BOUNDED THEN WE CAN CONCLUDE THAT MANY INVARIANTS OF VAR'S $\in \mathfrak{D}$ CAN ONLY ATTAIN FINITELY MANY VALUES.

THEOREM [VERDIER] LET $h : \mathcal{X} \longrightarrow T$ BE A PROPER MORPHISM OF FINITE TYPE VARIETIES.
$\exists\, U \subseteq T$ ZARISKI OPEN $(\neq \emptyset)$ SUCH $\mathcal{X}|_U \xrightarrow{h|_U} U$ IS A TOPOLOGICALLY TRIVIAL FIBRATION (IN THE EUCLIDEAN TOP.)



X_f
X_g
X_h
X_p
h
p
f
g

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COROLLARY LET $\mathfrak{D}$ BE A BOUNDED COLLECTION OF SMOOTH PROJECTIVE VAR'S. THEN $\forall\, p, q \geq 0$ $\exists\, M_{p,q} = M_{p,q}(\mathfrak{D}) \geq 0$ s.t. $h^{p,q}(X) \leq M_{p,q}$ $\forall\, X \in \mathfrak{D}$.

WHY IS BOUNDEDNESS AN INTERESTING PROPERTY?

RECALL THAT WE SAID YESTERDAY THAT 1 OF THE GOALS OF THE CLASSIFICATION OF ALG. VAR'S IS TO BUILD MODULI SPACES FOR

LOG CANONICAL MODELS

K-TRIVIAL PAIRS

LOG FANO PAIRS.

3 STEP (NON-SENSE) RECIPE FOR
A PROPER MODULI SPACE OF FINITE TYPE

1. Need to check that we are not trying to parametrize too many varieties! Key word: Boundedness
2. Need to choose what kind of degenerations will be admitted for varieties in $\mathfrak{D}$. Key word: Functor
3. Need to choose a way to construct the moduli space.
   Key word: Quotient
   Many available techniques: GIT, VGIT, KSBA, BB, ...

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THE EXAMPLES WE SAW EARLIER TELL US (UNSURPRISINGLY) THAT WE WILL NEED TO FIX SOME INVARIANTS IF WE WANT TO HAVE ANY HOPE OF PROVING BOUNDEDNESS.