BOUNDEDNESS QUESTIONS FOR CALABI-YAU'S

1. GENTLE INTRO TO THE MMP

2. BOUNDEDNESS

3. – 4. BOUNDEDNESS for CY's (elliptic)

+ log CY pairs

MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION

ONE OF THE MAIN GOALS OF ALGEBRAIC GEOMETRY IS TO PRODUCE A COMPLETE CLASSIFICATION OF PROJECTIVE VARIETIES.

MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION

ONE OF THE MAIN GOALS OF ALGEBRAIC GEOMETRY IS TO PRODUCE A COMPLETE CLASSIFICATION OF PROJECTIVE VARIETIES. TO THIS END, WE CAN PARTITION VARIETIES INTO EQUIVALENCE CLASSES USING 2 POSSIBLE EQUIVALENCE RELATIONS :

① ISOMORPHISM : $X \underset{f^{-1}}{\overset{f}{\rightleftarrows}} Y$

② BIRATIONAL ISOMORPHISM : $X \underset{g^{-1}}{\overset{g}{\dashrightarrow\dashleftarrow}} Y$ $\quad g, g^{-1}$ are rational maps.

$\mathbb{C}(X) =$ field of rat'l functions on X

MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION

X SMOOTH PROJECTIVE VARIETY $/ \mathbb{C}$

CANONICAL BUNDLE: $\det(\Omega^1_X)$

CANONICAL DIVISOR: Any Weil divisor K_X s.t. $\mathcal{O}_X(K_X) \cong$ canonical bundle.

MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION

X SMOOTH PROJECTIVE VARIETY $/ \mathbb{C}$

CONJECTURE STARTING FROM X, THERE EXISTS AN ALGORITHMIC WAY OF CONSTRUCTING A DIAGRAM OF THE FOLLOWING FORM

$$\begin{tikzcd} X \arrow[r, dashed, "\text{special birat'l}", "\text{transformations}"'] & X' \arrow[d, "f"] \\ & Z \end{tikzcd}$$

special morphism

SUCH THAT f SATISFIES 1 OF THE FOLLOWING:

1. f is BIRATIONAL
   K_Z is ample — Z is a can model.
2. f is a FIBRATION
   & $K_{X'} = f^* H$ ample on Z
   — the fibers will be K-trivial var's
3. f is a FIBR. & $-K_{X'}|_F$ is ample on F
   — the fibers will be Fano

MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION

HENCE IN THE BIRATIONAL CLASSIFICATION OF ALGEBRAIC VARIETIES WE HAVE 3 IMPORTANT CLASSES OF VARIETIES THAT ACT AS BUILDING BLOCKS

1. CANONICAL MODELS (K is ample)
2. K-TRIVIAL VAR'S ($K \equiv 0$)
3. FANO VAR'S ($-K$ is ample)

MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION

THE 2nd PART OF THE CLASSIFICATION AIMS THEN TO CLASSIFY THOSE VARIETIES THAT BELONG TO ONE OF THE 3 BUILDING BLOCKS

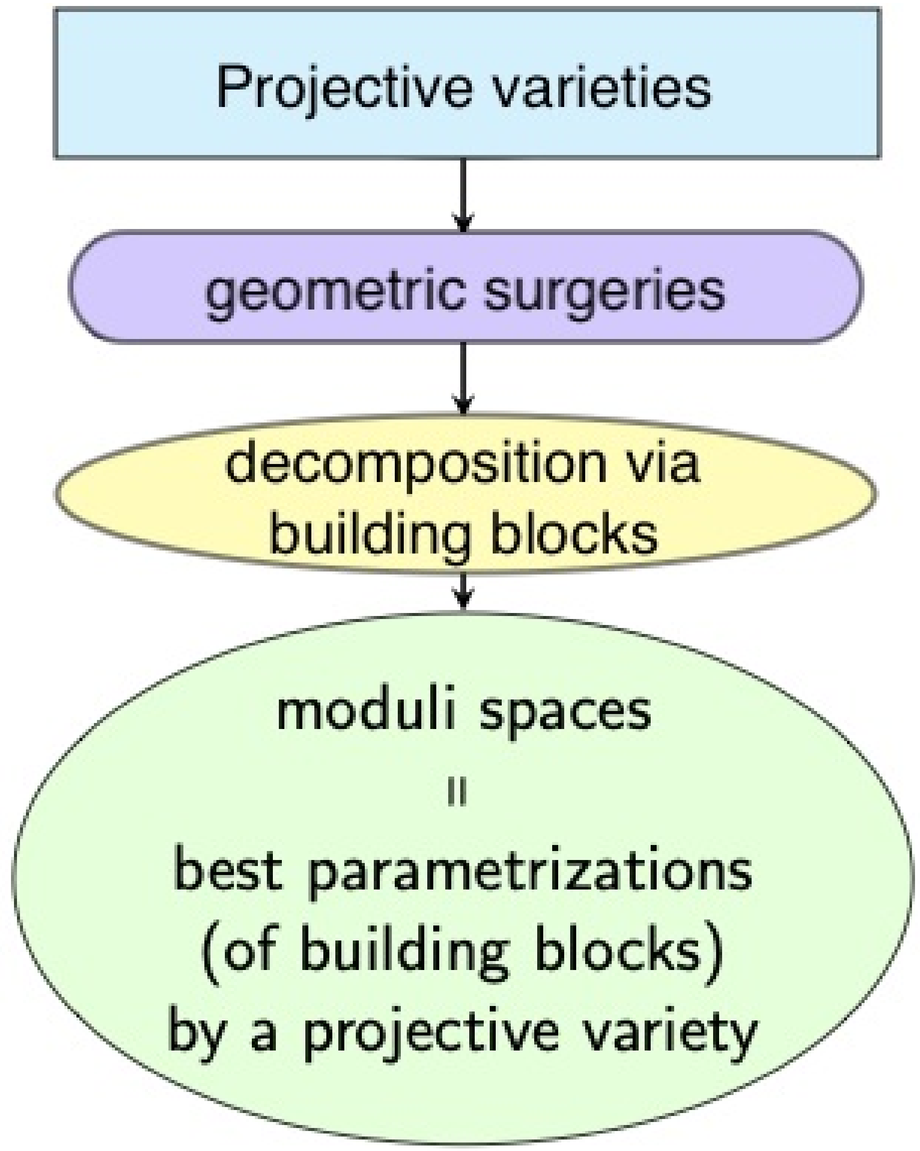
1. CANONICAL MODELS (K IS AMPLE)

2. K-TRIVIAL VAR'S ($K \equiv 0$)

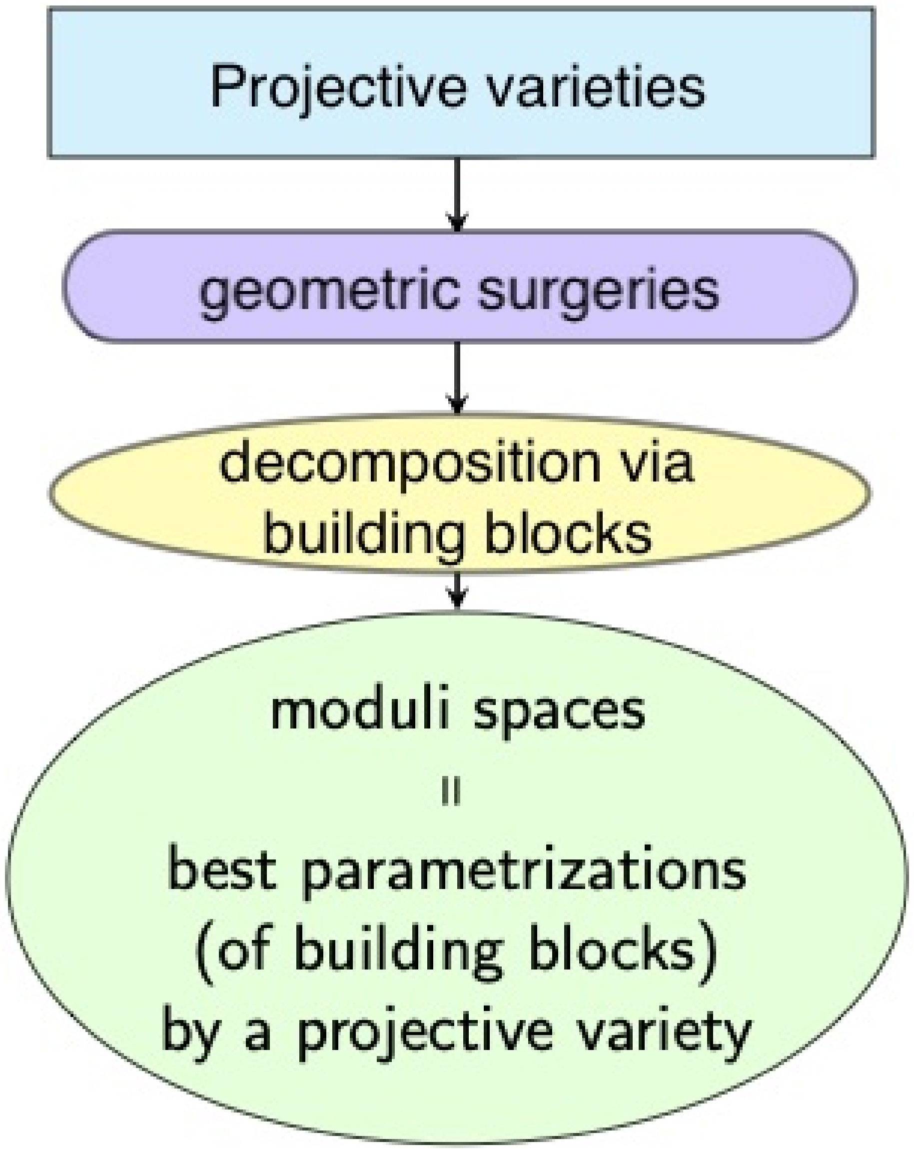
3. FANO VAR'S ($-K$ IS AMPLE)

MODULI SPACES :

MINIMAL MODEL PROGRAM: A GENTLE INTRODUCTION



MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION



IN ORDER TO COMPLETE SUCH CLASSIFICATION SCHEME, WE NEED ALSO TO WORK WITH SINGULAR VARIETIES.

MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION

X NORMAL PROJECTIVE VARIETY $/\mathbb{C}$

CANONICAL BUNDLE: $i_* \det(\Omega^1_{X^{sm}})$ rank 1-sheaf

$i : X^{sm} \hookrightarrow X$

CANONICAL DIVISOR: Zar. closure of any

$K_{X^{sm}}$

K_X will be a Weil divisor
(not. nec. Cartier)

MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION

DEFINITION ① A PAIR IS THE DATUM OF :

(i) X a normal variety

(ii) B an effective Weil divisor with coefficients $\in (0,1]$ BOUNDARY

② A LOG PAIR IS A PAIR (X,B) SUCH THAT

$K_X + B$ is $\mathbb{R}$-CARTIER (can be written as a $\mathbb{R}$-sum of Cartier divisors)

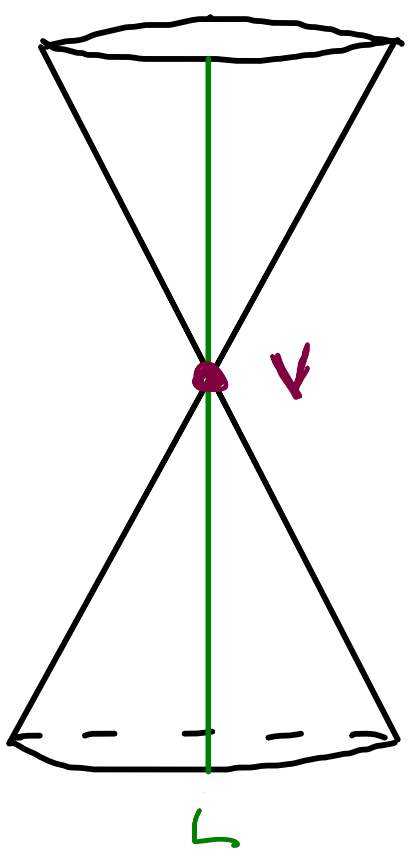
MINIMAL MODEL PROGRAM: A GENTLE INTRODUCTION

WHY IS IT USEFUL TO CONSIDER LOG PAIRS?

MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION

WHY IS IT USEFUL TO CONSIDER LOG PAIRS ?

CONSIDER FOR EXAMPLE



$C = V(x_0 x_1 - x_2^2) \subseteq \mathbb{P}^3$, $H \subseteq \mathbb{P}^3$ hyperpl.

C is SING. at v

$2L \in |H|_C|$ $\quad [\, 2L = V(x_0, x_0 x_1 - x_2^2)$

L IS NOT CARTIER AT v

ADJ. FORM: X SMOOTH
$D \overset{\cup\!\shortmid}{}$ SMOOTH codim 1

$$(K_X + D)\big|_D = K_D$$

MINIMAL MODEL PROGRAM: A GENTLE INTRODUCTION

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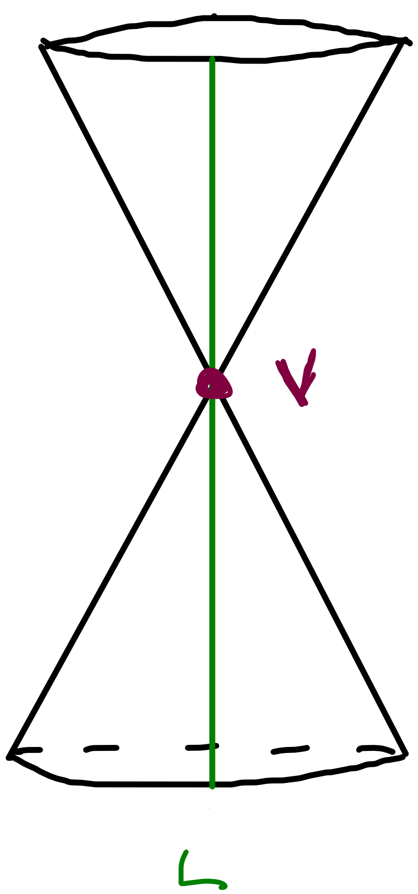

$C = V(x_0 x_1 - x_2^2) \subseteq \mathbb{P}^3$

$H \subseteq \mathbb{P}^3$ hyperpl.

$2L \sim H|_C$

ADJ. FORM: X SMOOTH, $D \subseteq X$ SMOOTH codim 1

$$(K_X + D)|_D = K_D$$

$$\left(K_{C \setminus \{v\}} + L|_{C\setminus\{v\}}\right)\Big|_{L|_{C\setminus\{v\}}} = K_{L|_{C\setminus\{v\}}}$$

$$(K_C + L)\cdot L = (-2H|_C + \tfrac{1}{2}H|_C)\cdot L = -\frac{3}{2} \implies (K_C + L)|_L = K_L + \frac{1}{2}v$$

MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION

WHY IS IT USEFUL TO CONSIDER LOG PAIRS?

CONSIDER A FIBRATION IN K-TRIVIAL VARIETIES

$$f : Y \longrightarrow X$$

MINIMAL MODEL PROGRAM: A GENTLE INTRODUCTION

WE SAW THAT WE WILL NEED TO CONSIDER SINGULAR OBJECTS.

TO DO THAT, WE NEED TO MEASURE THEIR SINGULARITIES.

WE WILL DO SO USING (BIRATIONAL) RESOLUTIONS.

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DEFINITION LET (X,B) BE A PAIR. A BIRAT'L MORPHISM

$$\pi : X' \longrightarrow X$$

IS A LOG RESOLUTION WHEN :

1. π IS A PROPER RESOLUTION OF X
2. $\tilde{B} + \mathrm{Exc}(\pi)$ is SIMPLE NORMAL CROSSING (S N C)

strict transform of B on X'

$\forall x' \in X$ locally analytically

$$(X', \tilde{B} + \mathrm{Exc}(\pi)) \overset{\mathrm{supp}}{\cong} \left(\mathbb{C}^n, \sum_{i=1}^{K} \{z_i = 0\}\right)$$

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1. X' IS SMOOTH;
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THEOREM [HIRONAKA] LOG RESOLUTION EXIST IN CHAR 0.

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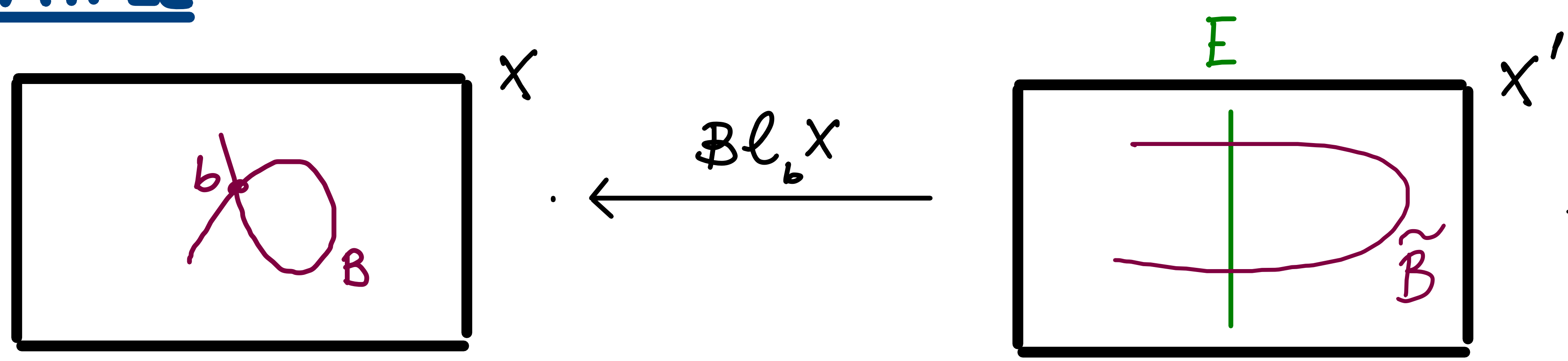

log resol.

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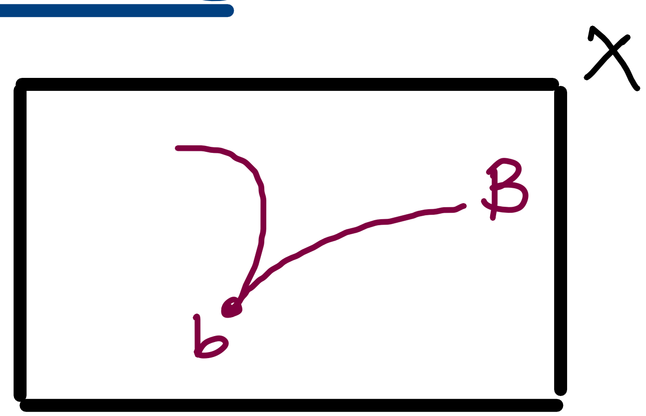

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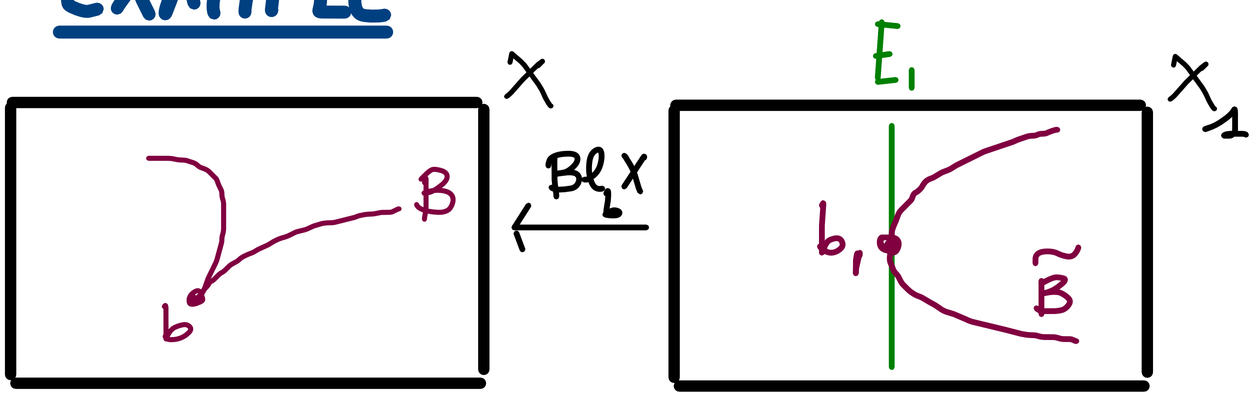

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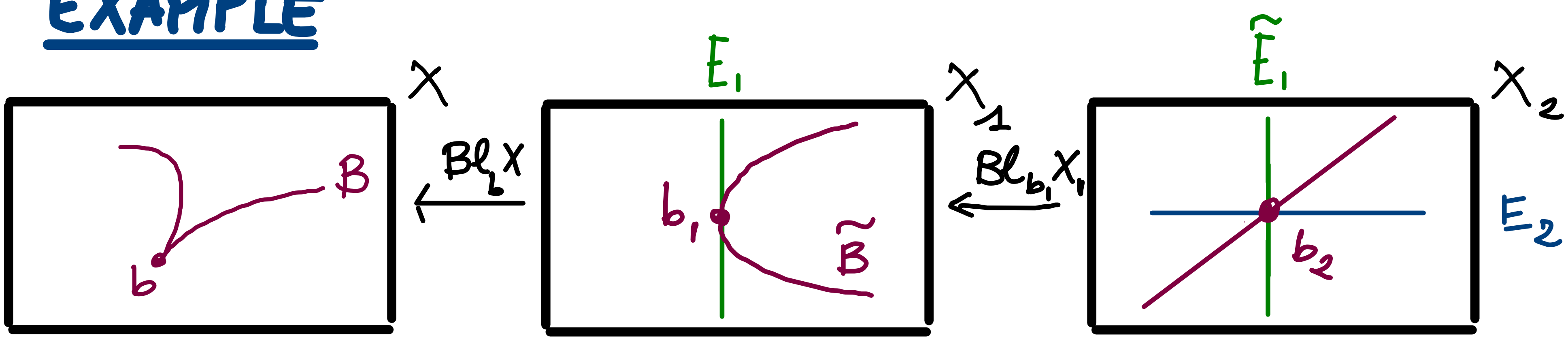

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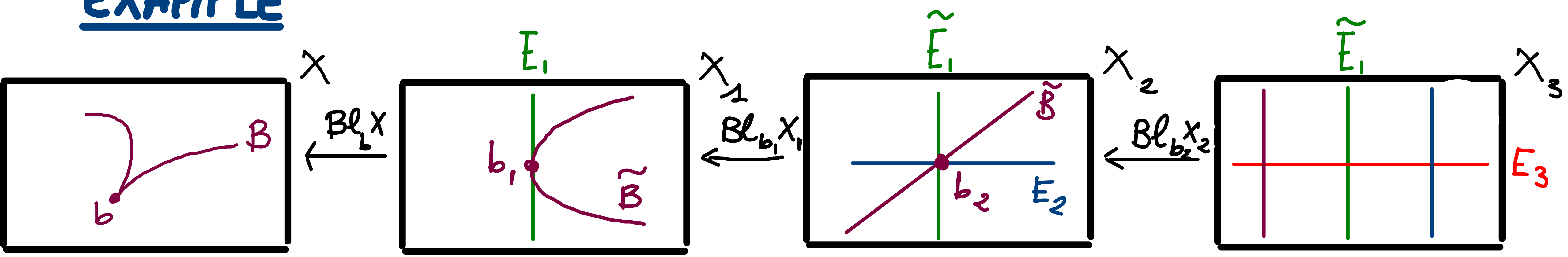

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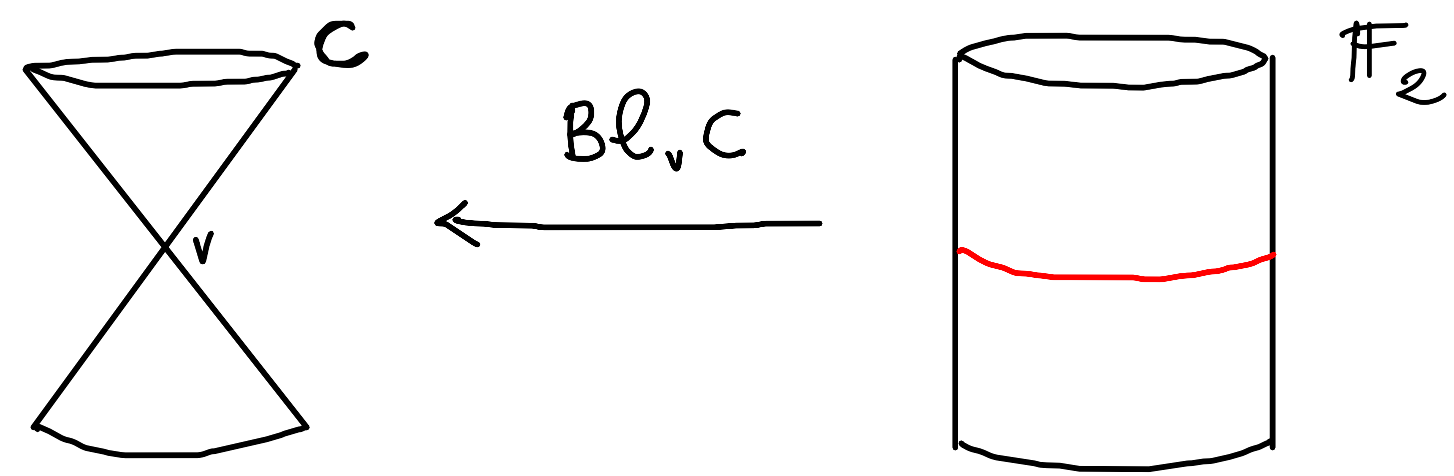

MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION

WE SAW THAT WE WILL NEED TO CONSIDER SINGULAR OBJECTS.
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WE WILL DO SO USING (BIRATIONAL) RESOLUTIONS.

GIVEN (X, B) A LOG PAIR AND A LOG RESOL.

$$\pi : X' \longrightarrow X$$

WE CAN ALWAYS WRITE IN A UNIQUE WAY

$$K_{X'} + \tilde{B} = \pi^*(K_X + B) + \sum_{\substack{\text{exc} \\ \text{div.}}} a_i E_i$$

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$$K_{X'} + B' = \pi^*(K_X + B)$$

WHERE $B' = \pi^{-1}_* B + \sum_{\substack{\text{exc.}\\ \text{div.}}} a_i E_i$

DEFINITION LET $D \subseteq X'$ BE A PRIME DIVISOR, THEN

$$a(D; X, B) := 1 - \text{coeff. of } D \text{ in } B'$$

log discrepancy

MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION

GIVEN (X,B) A LOG PAIR AND A LOG RESOL.

$$\pi : X' \longrightarrow X$$

WE CAN ALWAYS WRITE IN A UNIQUE WAY

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DEFINITION ① LET $D \subseteq X'$ BE A PRIME DIVISOR, THEN

$$a(D; X, B) := 1 - \text{coeff. of } D \text{ in } B'.$$

[THIS IS CALLED THE LOG DISCREPANCY OF D wrt (X,B)]

② THE TOTAL LOG DISCREPANCY OF (X,B) is

$$a(X,B) := \text{INF } a(D; X, B)$$

MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION

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② THE TOTAL LOG DISCREPANCY OF (X,B) is

$$a(X,B) := \text{INF } a(D; X, B)$$

SPECIAL DICHOTOMY : $a(X,B) = -\infty$

OR

$a(X,B) \geq 0$ MMP :)

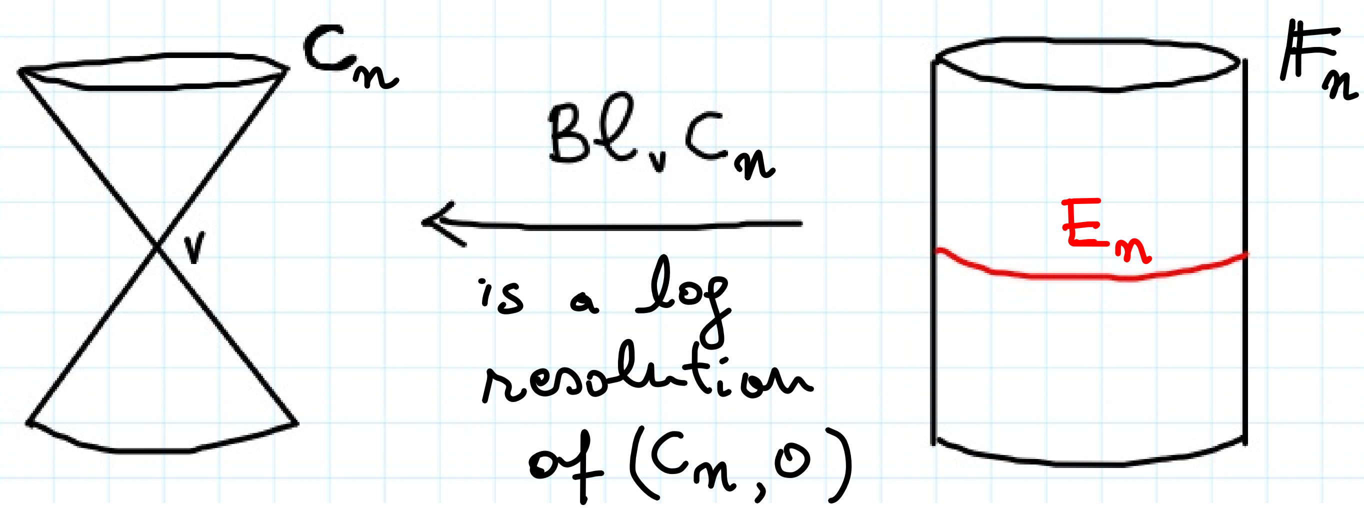
MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION

DEFINITION LET (X, B) BE A LOG PAIR.

THEN (X, B) IS

LOG CANONICAL		≥ 0
KLT (KAWAMATA LOG TERM.)	IF $a(X, B)$	> 0
CANONICAL		≥ 1
TERMINAL		> 1

EXAMPLE



C_n = CONE OVER RAT'L NORMAL CURVE OF deg n

LET'S COMPUTE THE LOG DISCREPANCY OF E_n

$$K_{\mathbb{F}_n} = \pi^*(K_{C_n}) + a_n E_n$$

$\Big\downarrow$ $+E_n$ on both sides

$$K_{\mathbb{F}_n} + E_n = \pi^*(K_{C_n}) + \underbrace{(1-a_n)}_{= a(E_n;\, C_n,\, 0)} E_n$$

$\Big\downarrow$ $\cdot E_n$ on both sides

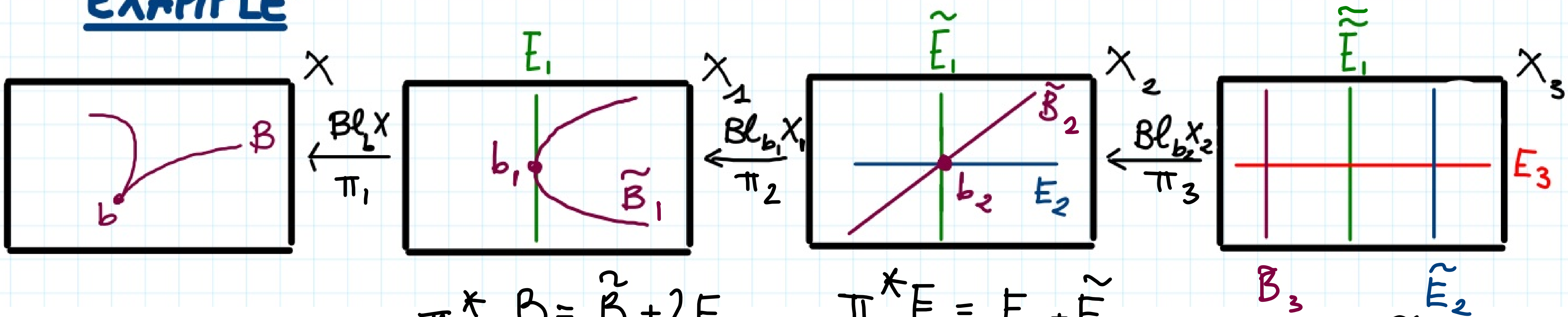
adj. to $E_n \cong \mathbb{P}^1$

$$-2 = (K_{\mathbb{F}_n} + E_n)\cdot E_n = \left(\pi^*(K_{C_n}) + a(E_n, C_n, 0) E_n\right)\cdot E_n = 0 + a(E_n, C_n, 0)(-n)$$

$$\Uparrow$$

$$a(E_n, C_n, 0) = \frac{2}{n}$$

EXAMPLE



$$\pi_1^* B = \tilde{B}_1 + 2E_1$$

$$\pi_1^* K_X = K_{X_1} - E_1$$

$$\Downarrow$$

$$\pi_1^*(K_X + B) = K_{X_1} + \tilde{B} + E_1$$

(X, B) IS NOT LC

$$\pi_2^* E_1 = E_2 + \tilde{E}_1$$

$$\pi_2^* \tilde{B}_1 = \tilde{B}_2 + E_2$$

$$\pi_2^* K_{X_1} = K_{X_2} - E_2$$

$$\Downarrow$$

$$\pi_2^*(K_{X_1} + \tilde{B} + E_1) = K_{X_2} + \tilde{E}_1 + E_2 + \tilde{B}_2$$

$$\pi_3^* \tilde{E}_1 = \tilde{\tilde{E}}_1 + E_3$$

$$\pi_3^* \tilde{B}_2 = \tilde{B}_3 + E_3$$

$$\pi_3^* E_2 = \tilde{E}_2 + E_3$$

$$\pi_3^* K_{X_2} = K_{X_3} - E_3$$

$$\Downarrow$$

$$\pi^*(K_{X_2} + \tilde{E}_1 + E_2 + \tilde{B}_2) = K_{X_3} + \tilde{\tilde{E}}_1 + \tilde{E}_2 + \tilde{B}_3 + 2E_3$$

MINIMAL MODEL PROGRAM: A GENTLE INTRODUCTION

~~X SMOOTH PROJECTIVE VARIETY / $\mathbb{C}$~~ (X, B) LOG CANONICAL PAIR

CONJECTURE STARTING FROM (X, B), THERE EXISTS AN ALGORITHMIC WAY OF CONSTRUCTING A DIAGRAM OF THE FOLLOWING FORM

$$\begin{tikzcd}
X \arrow[r, dashed, "\text{composition of special}"', "\text{birat'l maps}"] & (X', B') \arrow[d, "f"] \\
& Z
\end{tikzcd}$$

SUCH THAT f SATISFIES 1 OF THE FOLLOWING:

1. f BIRAT'L & $\cancel{K_Z}$ is AMPLE
2. f FIBRATION & $K_{X'} + B' = f^* H$ ample
3. f FIBRATION & $-(K_{X'} + B')$ AMPLE on fibers

MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION

IN ORDER TO ACHIEVE SUCH RESULT, WE WILL TRY TO CONTROL & TUNE THE POSITIVITY/NEGATIVITY OF LOG DIVISORS.

(X, B) LOG PAIR $\qquad K_X + B$ is $\mathbb{R}$-Cartier

$\forall C \subseteq X$ proper curve

$$(K_X + B) \cdot C$$

$K_X+B \cdot C \geq 0$
$\forall C \subseteq X$ proper curve

GOAL: either make $K_X + B$ nef after many birational transformations

OR

die trying (but in finite time)

THE CONE THEOREM

THEOREM [Mori, Kollár, Shokurov, ...]

LET (X, B) BE A LOG CANONICAL PAIR.

WE HAVE THE FOLLOWING DECOMPOSITION

$$\overline{NE}(X) = \overline{NE}(X)_{K_X+B \geq 0} + \overline{NE}(X)_{K_X+B < 0}$$

the closure the cone of effective curves

THE CONE THEOREM

THEOREM

LET (X,B) BE A LOG CANONICAL PAIR.

WE HAVE THE FOLLOWING DECOMPOSITION

$$\overline{NE}(X) = \overline{NE}(X)_{K_X+B\geq 0} + \overline{NE}(X)_{K_X+B<0} =$$

$$= \overline{NE}(X)_{K_X+B\geq 0} + \sum_{i\in I} \mathbb{R}_+ [C_i]$$

extremal rays

locally discrete away from $(K_X+B)^\perp$

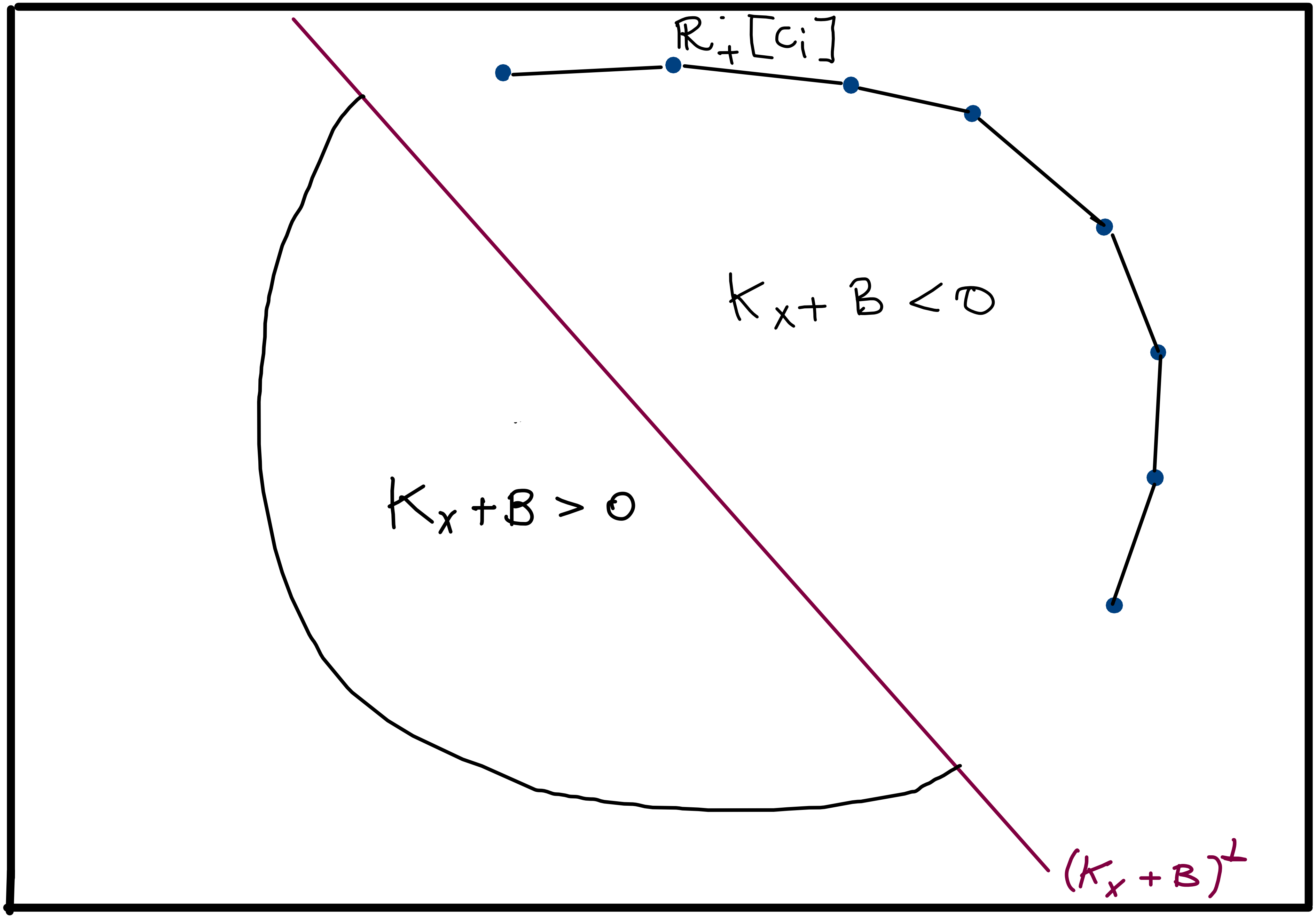
countable set

rat'l curves

$$-2\dim X \leq (K_X+B)\cdot C_i < 0$$

HORIZONTAL SLICE OF $\overline{NE}(X)$

extremal rays $\stackrel{\text{def}}{\equiv}$ $v_1 + v_2 \in R_i^- \Rightarrow v_1, v_2 \in R_i^-$



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LET (X, B) BE A LOG CANONICAL PAIR.

WE HAVE THE FOLLOWING DECOMPOSITION

$$\overline{NE}(X) = \overline{NE}(X)_{K_X+B \geq 0} + \overline{NE}(X)_{K_X+B<0} =$$

$$= \overline{NE}(X)_{K_X+B \geq 0} + \sum_{i \in I} \overbrace{\mathbb{R}_+ [C_i]}^{R_i \text{ extremal ray}}$$

$i \in I$: at most countable set

$[C_i]$: rat'l curves

$-2\dim X \leq (K_X + B) \cdot C_i < 0$

MOREOVER, $\forall i \in I$, $\exists$ $\mathrm{cont}_{R_i} : X \longrightarrow Z_i$ SUCH THAT cont_{R_i} HAS CONNECTED FIBERS & IF $\mathrm{cont}_{R_i}(C) = pt \Rightarrow [C] \in R_i$.